Analysis on Vector Bundles over Noncommutative Tori

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ABSTRACT

Noncommutative geometry is the study of noncommutative algebras, especially C^*-algebras, and their geometric interpretation as topological spaces. One C^*-algebra particularly important in physics is the noncommutative n-torus, the irrational rotation C^*-algebra A_Θ with n unitary generators $U_1, \ldots, U_n$ which satisfy $U_k U_j = e^{2\pi i \theta_{j,k}} U_j U_k$ and $U_j^* = U_j^{-1}$, where $\Theta \in M_n(\mathbb{R})$ is skew-symmetric with upper triangular entries that are irrational and linearly independent over $\mathbb{Q}$. We focus on two projects: an analytically detailed derivation of the pseudodifferential calculus on noncommutative tori, and a proof of an index theorem for vector bundles over the noncommutative two torus. We use Raymond's definition of an oscillatory integral with Connes' construction of pseudodifferential operators to rederive the calculus in more detail, following the strategy of the derivations in Wong's book on pseudodifferential operators. We then define the corresponding analog of Sobolev spaces on noncommutative tori, for which we prove analogs of the Sobolev and Rellich lemmas, and extend all of these results to vector bundles over noncommutative tori. We extend Connes and Tretkoff's analog of the Gauss–Bonnet theorem for the noncommutative two torus to an analog of the McKean–Singer index theorem for vector bundles over the noncommutative two torus, proving a rearrangement lemma where a self-adjoint idempotent e appears in the denominator but does not commute with the k^2 already there from the rearrangement lemma proven by Connes and Tretkoff.

PUBLISHED CONTENT AND CONTRIBUTIONS

(a) Farzad Fathizadeh, Franz Luef, and Jim Tao. A twisted local index formula for curved noncommutative two tori. *arXiv e-prints*, art. arXiv:1904.03810, Apr 2019.
J. T. contributed new rearrangement lemmas, proven using techniques from analytic combinatorics, and Mathematica code that used Mathematica's symbolic pattern language to automate lengthy, complicated index theory calculations.

(b) Jim Tao. The theory of pseudo-differential operators on the noncommutative n-torus. In *Journal of Physics Conference Series*, volume 965 of *Journal of Physics Conference Series*, page 012042, February 2018. doi: 10.1088/1742-6596/965/1/012042.
J. T. participated in proving the results and writing this paper.

(c) Jim Tao. The theory of pseudo-differential operators on the noncommutative n-torus. *arXiv e-prints*, art. arXiv:1704.02507, Apr 2017.
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Chapter 1

THE PSEUDO-DIFFERENTIAL CALCULUS ON NONCOMMUTATIVE TORI

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1.1 Introduction

The methods of spectral geometry are useful for investigating the metric aspects of noncommutative geometry [11, 14, 16, 20] and in these contexts require extensive use of pseudo-differential operators. In the foundational paper [13], Connes showed that, by direct analogy with the theory [38, 61, 69] of pseudo-differential operators on $\mathbb{R}^n$, one may derive a similar pseudo-differential calculus on noncommutative n tori $\mathbb{T}^n_\theta$. With the development of this calculus came many results concerning the local differential geometry of noncommutative tori for $n = 2, 4$, as shown in the groundbreaking paper [23] in which the Gauss–Bonnet theorem on $\mathbb{T}^2_\theta$ is proved and later papers [22, 30–33]. In these papers, the flat geometry of $\mathbb{T}^n_\theta$, which was studied in [13], is conformally perturbed using a Weyl factor given by a positive invertible smooth element in $C^\infty(\mathbb{T}^n_\theta)$. Connes' pseudo-differential calculus is critically used to apply heat kernel techniques to geometric operators on $\mathbb{T}^n_\theta$ to derive small time heat kernel expansions that encode local geometric information such as scalar curvature. As discovered in [22, 32, 33], a purely noncommutative feature that appears in the computations and in the final formula for the curvature is the modular automorphism of the state implementing the conformal perturbation of the metric.

Certain details of the proofs in the derivation of the calculus in [13] were omitted, such as the evaluation of oscillatory integrals, so we make it the objective of this paper to fill in all the details. After reproving in more detail the formula for the symbol of the adjoint of a pseudo-differential operator and the formula for the symbol of a product of two pseudo-differential operators, we define the corresponding analog of Sobolev spaces for which we prove the Sobolev and Rellich lemmas. We then extend these results to finitely generated projective right modules over the noncommutative n torus.

We list these results below.

Theorem 1.1.1. *Suppose P is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M. Then the symbol of the adjoint P^* is of order M and satisfies $\sigma(P^*) \sim \sum_{\ell \in \mathbb{Z}^n_{\geq 0}} \frac{\partial^\ell \delta^\ell [(\rho(\xi))^*]}{\ell_1! \cdots \ell_n!}$.*

Theorem 1.1.2. *Suppose that P is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M_1, and Q is a pseudo-differential operator with symbol $\sigma(Q) = \phi = \phi(\xi)$ of order M_2. Then the symbol of the product QP is of order $M_1 + M_2$ and satisfies $\sigma(QP) \sim \sum_{\ell \in \mathbb{Z}^n_{\geq 0}} \frac{1}{\ell_1! \cdots \ell_n!} \partial^\ell \phi(\xi) \delta^\ell \rho(\xi)$, where $\partial^\ell := \prod_j \partial_j^{\ell_j}$ and $\delta^\ell := \prod_j \delta_j^{\ell_j}$.*

Theorem 1.1.3. *For $s > k + 1$, $H^s \subseteq A_\theta^k$.*

Theorem 1.1.4. *Let $\{a_N\} \in A_\theta^\infty$ be a sequence. Suppose that there is a constant C so that $||a_N||_s \leq C$ for all N. Let $s > t$. Then there is a subsequence $\{a_{N_j}\}$ that* *converges in H^t.*

Theorem 1.1.5. *(a) For a pseudo-differential operator P with $r \times r$ matrix valued symbol $\sigma(P) = \rho = \rho(\xi)$, the symbol of the adjoint P^* satisfies*

$$\sigma(P^*) \sim \sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq 0})^n} \frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho(\xi))^*}{\ell_1!\cdots\ell_n!}.$$

(b) If Q is a pseudo-differential operator with $r \times r$ matrix valued symbol $\sigma(Q) = \rho' = \rho'(\xi)$, then the product PQ is also a pseudo-differential operator and has symbol

$$\sigma(PQ) \sim \sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq 0})^n} \frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}(\rho(\xi))\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho'(\xi))}{\ell_1!\cdots\ell_n!}.$$

Theorem 1.1.6. *For $s > k+1$, $H^s \subseteq e(A_\theta^k)^r$.*

Theorem 1.1.7. *Let $\{\vec{a}_N\} \in e(A_\theta^\infty)^r$ be a sequence. Suppose that there is a constant C so that $||\vec{a}_N||_s \leq C$ for all N. Let $s > t$. Then there is a subsequence $\{\vec{a}_{N_j}\}$ that converges in H^t.*

1.2 Preliminaries

Fix some skew symmetric $n \times n$ matrix θ with upper triangular entries in $\mathbb{R}\backslash\mathbb{Q}$ that are linearly independent over $\mathbb{Q}$. Consider the irrational rotation C^*-algebra A_θ with n unitary generators $U_1, \ldots, U_n$ which satisfy $U_k U_j = e^{2\pi i\theta_{j,k}} U_j U_k$ and $U_j^* = U_j^{-1}$ (A_θ is the universal such C^*-algebra with these generators and relations, so we opt not to use a GNS construction). Let $\{\alpha_s\}_{s\in\mathbb{R}^n}$ be a n-parameter group of automorphisms given by $\prod_j U_j^{m_j} \mapsto e^{is\cdot m} \prod_j U_j^{m_j}$. We define the subalgebra A_θ^k of C^k elements of A_θ to be those $a \in A_\theta$ such that the mapping $\mathbb{R}^n \to A_\theta$ given by $s \mapsto \alpha_s(a)$ is C^k, and we define the subalgebra A_θ^∞ of smooth elements of A_θ to be those $a \in A_\theta$ such that the mapping $\mathbb{R}^n \to A_\theta$ given by $s \mapsto \alpha_s(a)$ is smooth. An alternative definition of the subalgebra A_θ^∞ of smooth elements is the elements in A_θ that can be expressed by an expansion of the form $\sum_{m\in\mathbb{Z}^n} a_m \prod_j U_j^{m_j}$, where the sequence $\{a_m\}_{m\in\mathbb{Z}^n}$ is in the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ in the sense that, for all $\alpha \in \mathbb{Z}^n$,

$$\sup_{m\in\mathbb{Z}^n} (\prod_j |m_j|^{\alpha_j} |a_m|) < \infty.$$

Define the trace $\tau : A_\theta \to \mathbb{C}$ by $\tau(\prod_j U_j^{m_j}) = 0$ for m_j not all zero and $\tau(1) = 1$ and define an inner product $\langle \cdot, \cdot \rangle : A_\theta \times A_\theta \to \mathbb{C}$ by $\langle a, b \rangle = \tau(b^* a)$ with induced norm $|| \cdot || : A_\theta \to \mathbb{R}_{\geq 0}$. Let $D_j = -i\partial_j$ and define derivations δ_j by the relations $\delta_j(U_j) = U_j$ and $\delta_j(U_k) = 0$ for $j \neq k$. For convenience, denote $\partial^\ell := \prod_j \partial_j^{\ell_j}$, $\delta^\ell := \prod_j \delta_j^{\ell_j}$, and $D^\ell := \prod_j D_j^{\ell_j}$. We define a map $\psi : \rho \mapsto P_\rho$ assigning a pseudo-differential operator on A_θ^∞ to a symbol $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$.

Definition 1.2.1. For $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$, let P_ρ be the pseudo-differential operator sending arbitrary $a \in A_\theta^\infty$ to

$$P_\rho(a) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \rho(\xi) \alpha_s(a) \, \mathrm{d}s \, \mathrm{d}\xi.$$

The integral above does not converge absolutely; it is an oscillatory integral. We define oscillatory integrals below as in [61].

Definition 1.2.2. Let q be a nondegenerate real quadratic form on $\mathbb{R}^n$, a be a C^∞ complex-valued function defined on $\mathbb{R}^n$ such that the functions $(1 + |x|^2)^{-m/2} \partial^\alpha a(x)$ are bounded on $\mathbb{R}^n$ for all $\alpha \in \mathbb{Z}_{\geq 0}^n$, and φ be a Schwartz function, i.e. the functions

$x^\alpha \partial^\beta \varphi(x)$ are bounded on $\mathbb{R}^n$ for all pairs $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. Suppose further that $\varphi(0) = 1$. Then the limit

$$\lim_{\epsilon \to 0} \int e^{iq(x)} a(x) \varphi(\epsilon x) \, \mathrm{d}x$$

exists, is independent of φ (as long as $\varphi(0) = 1$), and is equal to $\int e^{iq(x)} a(x) \, \mathrm{d}x$ when $a \in L^1$. When $a \notin L^1$, we continue to denote this limit by $\int e^{iq(x)} a(x) \, \mathrm{d}x$, and have an estimate

$$\left| \int e^{iq(x)} a(x) \, \mathrm{d}x \right| \leq C_{q,m} \max_{|\alpha| \leq m+n+1} \inf\{U \in \mathbb{R} : |(1 + |x|^2)^{-m/2} \partial^\alpha a| \leq U \text{ a.e.}\}$$

where $C_{q,m}$ depends only on the quadratic form q and the order m.

As shown in [61], oscillatory integrals behave essentially like absolutely convergent integrals in that one can still make changes of variables, integrate by parts, differentiate under the integral sign, and interverse integral signs. Given certain conditions on ρ, $P_\rho(a)$ satisfies the conditions of the oscillatory integral, and we can evaluate $P_\rho(a)$.

Definition 1.2.3. An element $\rho = \rho(\xi) = \rho(\xi_1, \ldots, \xi_n)$ of $C^\infty(\mathbb{R}^n, A^\infty_\theta)$ is a symbol of order m if and only if for all non-negative integers $i_1, \ldots, i_n, j_1, \ldots, j_n$

$$||\delta_1^{i_1} \cdots \delta_n^{i_n} (\partial_1^{j_1} \cdots \partial_n^{j_n} \rho)(\xi)|| \leq C_\rho (1 + |\xi|)^{m - |j|}.$$

Example 2.6(i) of [61] gives a convenient formula for evaluating the oscillatory integrals that appear in our calculation of $P_\rho(a)$.

Proposition 1.2.4. *Suppose that, for some m, a is a C^∞ complex-valued function defined on $\mathbb{R}^n$ such that the functions $(1 + |x|^2)^{-m/2} \partial^\alpha a(x)$ are bounded on $\mathbb{R}^n$ for* *all $\alpha \in \mathbb{Z}^n_{\geq 0}$. Then*

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy\cdot\eta} a(y) \, \mathrm{d}y \, \mathrm{d}\eta = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy\cdot\eta} a(\eta) \, \mathrm{d}y \, \mathrm{d}\eta = a(0).$$

We apply Proposition 1.2.4 to get a basic result.

Lemma 1.2.5. *Let $a = \sum_{m\in\mathbb{Z}^n} a_m \prod_j U_j^{m_j}$ be an arbitrary element of A_θ^∞ and let $\rho \in C^\infty(\mathbb{R}^n, A_\theta^\infty)$ be a symbol of order M. Then*

$$P_\rho(a) = \sum_{m\in\mathbb{Z}^n} \rho(m) a_m \prod_{j=1}^n U_j^{m_j}.$$

Proof. First consider the case $a = \prod_j U_j^{m_j}$. We get

$$\begin{aligned}
P_\rho\left(\prod_{j=1}^n U_j^{m_j}\right) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \rho(\xi) \alpha_s \left(\prod_{j=1}^n U_j^{m_j}\right) \mathrm{d}s \, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \rho(\xi) e^{is\cdot m} \prod_{j=1}^n U_j^{m_j} \, \mathrm{d}s \, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot(\xi - m)} \rho(\xi) \, \mathrm{d}s \, \mathrm{d}\xi \prod_{j=1}^n U_j^{m_j} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\eta} \rho(\eta + m) \, \mathrm{d}s \, \mathrm{d}\eta \prod_{j=1}^n U_j^{m_j} \\
&= \rho(m) \prod_{j=1}^n U_j^{m_j},
\end{aligned}$$

as desired, having substituted $\eta = \xi - m$ and applied the result of Proposition 1.2.4. Now consider the general case $a = \sum_{m\in\mathbb{Z}^n} a_m \prod_j U_j^{m_j}$. Since α_s is an automorphism on A_θ, we get

$$P_\rho(a) = \sum_{m\in\mathbb{Z}^n} \rho(m) a_m \prod_{j=1}^n U_j^{m_j},$$

and we are done. □

1.3 Asymptotic formula for the symbol of the adjoint of a pseudo-differential operator

Here we prove the formula for the symbol of the adjoint for the noncommutative n torus, adapting the proof of Lemma 1.2.3 of [38] to the noncommutative n torus.

Theorem 1.3.1. *Suppose P is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M. Then the symbol of the adjoint P^* is of order M and satisfies*

$$\sigma(P^*) \sim \sum_{\ell \in \mathbb{Z}^n_{\geq 0}} \frac{\partial^\ell \delta^\ell [(\rho(\xi))^*]}{\ell_1! \cdots \ell_n!}.$$

Proof. Let $a, b \in A^\infty_\theta$. We have

$$\begin{aligned}
\langle P_\rho(a), b \rangle &= \tau\left(b^* \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \rho(\xi) \alpha_s(a)\, \mathrm{d}s\, \mathrm{d}\xi \right) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \tau(b^* \rho(\xi) \alpha_s(a))\, \mathrm{d}s\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \tau(\alpha_{-s}(\rho(\xi)^* b)^* a)\, \mathrm{d}s\, \mathrm{d}\xi \\
&= \tau\left(\left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\cdot\xi} \alpha_{-s}(\rho(\xi)^* b)\, \mathrm{d}s\, \mathrm{d}\xi \right)^* a \right) \\
&= \langle a, P^*_\rho(b) \rangle,
\end{aligned}$$

where

$$\begin{aligned}
P^*_\rho(b) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\cdot\xi} \alpha_{-s}(\rho(\xi)^* b)\, \mathrm{d}s\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\cdot\xi} \alpha_{-s}(\rho(\xi)^*) \alpha_{-s}(b)\, \mathrm{d}s\, \mathrm{d}\xi \\
&= \sum_{m,k} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{+is\cdot\xi} e^{+is\cdot m} \rho_m(\xi)^* e^{-is\cdot k} b_k\, \mathrm{d}s\, \mathrm{d}\xi\, U_n^{-m_n} \cdots U_1^{-m_1} U_1^{k_1} \cdots U_n^{k_n} \\
&= \sum_{m,k} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\eta} \rho_m((k-m) - \eta)^* b_k\, \mathrm{d}s\, \mathrm{d}\eta\, U_n^{-m_n} \cdots U_1^{-m_1} U_1^{k_1} \cdots U_n^{k_n}
\end{aligned}$$

$$= \sum_{m,k} \rho_m(k-m)^* b_k U_n^{-m_n} \cdots U_1^{-m_1} U_1^{k_1} \cdots U_n^{k_n}$$

$$= \sum_{m,k} (\rho_m(k-m) U_1^{m_1} \cdots U_n^{m_n})^* (b_k U_1^{k_1} \cdots U_n^{k_n}),$$

so

$$\begin{aligned}
\sigma(P_\rho^*)(\xi) &= \left[\sum_m \rho_m(\xi - m) \prod_{j=1}^n U_j^{m_j} \right]^* \\
&= \left[\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \sum_m \rho_m(\xi - y) \alpha_x \left(\prod_{j=1}^n U_j^{m_j} \right) \mathrm{d}x \, \mathrm{d}y \right]^* \\
&= \left[\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \alpha_x(\rho(\xi - y)) \, \mathrm{d}x \, \mathrm{d}y \right]^* .
\end{aligned}$$

We have

$$\begin{aligned}
\rho(\xi - y) &= \sum_{|\ell| < N_1} \frac{(-y)^\ell}{\ell!} (\partial^\ell \rho)(\xi) + R_{N_1}(\xi, y) \\
\alpha_x(\rho(\xi - y)) &= \sum_{|\ell| < N_1} \sum_m \frac{(-y)^\ell}{\ell!} (\partial^\ell \rho_m)(\xi) e^{ix\cdot m} \left(\prod_{j=1}^n U_j^{m_j} \right) + \alpha_x(R_{N_1}(\xi, y)),
\end{aligned}$$

where

$$R_{N_1}(\xi, y) = N_1 \sum_{|\ell| = N_1} \frac{(-y)^\ell}{\ell!} \int_0^1 (1-\gamma)^{N_1 - 1} (\partial^\ell \rho)(\xi - y\gamma) \, \mathrm{d}\gamma$$

so the corresponding symbol is

$$\begin{aligned}
\sigma(P_\rho^*)(\xi) &= \left[\sum_{|\ell| < N_1} \sum_m \frac{(-m)^\ell}{\ell!} (\partial^\ell \rho_m)(\xi) \prod_{j=1}^m U_j^{m_j} + T_{N_1}(\xi, y) \right]^* \\
&= \sum_{|\ell| < N_1} \frac{\partial^\ell \delta^\ell [(\rho(\xi))^*]}{\ell!} + T_{N_1}(\xi, y)^*,
\end{aligned}$$

where

$$T_{N_1}(\xi, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \alpha_x(R_{N_1}(\xi, y)) \, \mathrm{d}x \, \mathrm{d}y.$$

It remains to show that this symbol is of order M. Obviously,

$$\sum_{N\leq \ell<N_1} \frac{\partial^\ell \delta^\ell[(\rho(\xi))^*]}{\ell!} \in S^{M-N},$$

so we need to show that the remainder is of order $M-N_1$. Note that

$$\sigma(P_\rho^*)(\xi) - \sum_{|\ell|<N_1} \frac{\partial^\ell \delta^\ell[(\rho(\xi))^*]}{\ell!} = T_{N_1}(\xi, y)^*.$$

Integrating by parts, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\frac{(-y)^\ell}{\ell!}\alpha_x\left(\int_0^1 (1-\gamma)^{N_1-1}(\partial^\ell\rho)(\xi-y\gamma)\,\mathrm{d}\gamma\right)\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}(-D_x)^\ell\alpha_x\left(\int_0^1 (1-\gamma)^{N_1-1}(\partial^\ell\rho)(\xi-y\gamma)\,\mathrm{d}\gamma\right)\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}(-\delta)^\ell\alpha_x\left(\int_0^1 (1-\gamma)^{N_1-1}(\partial^\ell\rho)(\xi-y\gamma)\,\mathrm{d}\gamma\right)\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\int_0^1 (1-\gamma)^{N_1-1}(-\delta)^\ell\alpha_x((\partial^\ell\rho)(\xi-y\gamma))\,\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{\ell!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}(-1)^\ell\int_0^1 (1-\gamma)^{N_1-1}\alpha_x((\delta^\ell\partial^\ell\rho)(\xi-y\gamma))\,\mathrm{d}\gamma\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where, for arbitrary $a=\sum_m a_m\prod_j U_j^{m_j}$,

$$\begin{aligned}
(-D_x)^\ell\alpha_x(a) &= (-D_x)^\ell\alpha_x\left(\sum_m a_m\prod_{j=1}^n U_j^{m_j}\right)\\
&= (-D_x)^\ell\sum_m a_m e^{ix\cdot m}\prod_{j=1}^n U_j^{m_j}\\
&= \sum_m a_m(-m)^\ell e^{ix\cdot m}\prod_{j=1}^n U_j^{m_j}\\
&= (-\delta)^\ell\alpha_x(a).
\end{aligned}$$

Since $\rho\in S^M$ and $|\ell|=N_1$ we have $\delta^\ell\partial^\ell\rho\in S^{M-N_1}$. We get the boundedness of

$$\ell^{N_1-M}(\xi)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\alpha_x(R_{N_1}(\xi,y))\,\mathrm{d}x\,\mathrm{d}y$$

because $\partial^\ell R_{N_1}(y,\xi)$ is the rest of index N_1 in the Taylor expansion of $\partial^\ell(\xi-y\gamma)$ for which one has $\partial^\ell\rho\in S^{M-N_1}$. □

1.4 Asymptotic formula for the symbol of a product of two pseudo-differential operators

Next we prove the formula for the product or composition of symbols for the noncommutative n torus, adapting the proof of Theorem 7.1 of [69].

Theorem 1.4.1. *Suppose that P is a pseudo-differential operator with symbol $\sigma(P) = \rho = \rho(\xi)$ of order M_1, and Q is a pseudo-differential operator with symbol $\sigma(Q) = \phi = \phi(\xi)$ of order M_2. Then the symbol of the product QP is of order $M_1 + M_2$ and satisfies*

$$\sigma(QP) \sim \sum_{\ell \in \mathbb{Z}^n_{\geq 0}} \frac{1}{\ell_1! \cdots \ell_n!} \partial^\ell \phi(\xi) \delta^\ell \rho(\xi),$$

where $\partial^\ell := \prod_j \partial_j^{\ell_j}$ and $\delta^\ell := \prod_j \delta_j^{\ell_j}$.

Proof. We want to show that if $\rho : \mathbb{R}^n \to A_\theta^\infty$ is of order M_1 and $\phi : \mathbb{R}^n \to A_\theta^\infty$ is of order M_2, $P_\phi \circ P_\rho = P_\mu$, where μ is of order $M_1 + M_2$ and has asymptotic expansion

$$\mu \sim \sum_\ell \frac{1}{\ell!} \partial^\ell \phi(\xi) \delta^\ell \rho(\xi).$$

Let $\{\varphi_k\}$ be the partition of unity constructed in Theorem 6.1 of [69] and define $\phi_k(\xi) := \phi(\xi)\varphi_k(\xi)$. We have

$$P_{\phi_k}(a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{is\cdot\xi} \phi_k(\xi) \alpha_s(a) \, \mathrm{d}s \, \mathrm{d}\xi.$$

Summing over k from zero to infinity and applying Fubini's Theorem, we get

$$\begin{aligned}
\sum_{k=0}^\infty P_{\phi_k}(a) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \sum_{k=0}^\infty \phi_k(\xi) \alpha_s(a) \, \mathrm{d}s \, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-is\cdot\xi} \phi(\xi) \alpha_s(a) \, \mathrm{d}s \, \mathrm{d}\xi,
\end{aligned}$$

so

$$P_\phi(a) = \sum_{k=0}^{\infty} P_{\phi_k}(a)$$

and the convergence of the series is absolute and uniform for all $a \in A_\theta^\infty$. We want to compute the symbol of $P_\phi \circ P_\rho$, but issues with convergence of integrals make it so we need to compute the symbol of $P_{\phi_k} \circ P_\rho$. Let $a \in A_\theta^\infty$ be arbitrary. Applying $P_{\phi_k} \circ P_\rho$ we get

$$\begin{aligned}
P_{\phi_k}(P_\rho(a)) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi} \phi_k(\xi)\alpha_s(P_\rho(a))\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi} \phi_k(\xi)\alpha_s\left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-it\cdot\eta}\rho(\eta)\alpha_t(a)\,\mathrm{d}t\,\mathrm{d}\eta\right) \mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi-it\cdot\eta}\phi_k(\xi)\alpha_s(\rho(\eta))\alpha_{s+t}(a)\,\mathrm{d}t\,\mathrm{d}\eta\right\}\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi-i(y-x)\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\eta\right\}\mathrm{d}x\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-\eta)-iy\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\eta\right\}\mathrm{d}x\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma-iy\cdot\tau}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau\right\}\mathrm{d}x\,\mathrm{d}\sigma \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\,\mathrm{d}x\,\mathrm{d}\sigma\right\}\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\mu_k(\tau)\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau,
\end{aligned}$$

where

$$\mu_k(\tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau)))\,\mathrm{d}x\,\mathrm{d}\sigma,$$

having done the changes of variables $(x, y) = (s, s+t)$ and $(\sigma, \tau) = (\xi - \eta, \eta)$ and applied Proposition 1.2.4. This suggests that

$$P_\phi(P_\rho(a)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\mu(\tau)\alpha_y(a)\,\mathrm{d}y\,\mathrm{d}\tau,$$

where $\mu(\tau) = \sum_{k=0}^{\infty} \mu_k(\tau)$. We need to show that μ is a symbol in $S^{M_1+M_2}$ and has our desired asymptotic expression. Define μ_k by

$$\mu_k(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \phi_k(\xi + y) \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y$$

for all $\xi \in \mathbb{R}^n$. By Taylor's formula with integral remainder given in Theorem 6.3 of [69], we get

$$\phi_k(\xi + y) = \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi_k)(\xi) + R_{N_1}(y, \xi),$$

where

$$R_{N_1}(y, \xi) = N_1 \sum_{|\ell| = N_1} \frac{y^\ell}{\ell!} \int_0^1 (1 - \gamma)^{N_1 - 1} (\partial^\ell \phi_k)(\xi + \gamma y) \, \mathrm{d}\gamma \tag{1.1}$$

for all $y, \xi \in \mathbb{R}^2$. Substituting back into our expression for $\mu_k(\xi)$ we get

$$\mu_k(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi_k)(\xi) \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y + T_{N_1}^{(k)}(\xi),$$

where

$$T_{N_1}^{(k)}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} R_{N_1}(y, \xi) \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y.$$

Expressing $\rho(\xi)$ as $\rho(\xi) = \sum_m \rho_m(\xi) \prod_{j=1}^n U_j^{m_j}$, we see that

$$\alpha_x(\rho(\xi)) = \sum_m \rho_m(\xi) e^{ix\cdot m} \prod_{j=1}^n U_j^{m_j}$$

so

$$\begin{aligned}
\mu_k(\xi) - T_{N_1}^{(k)}(\xi) &= \sum_{|\ell| < N_1} \sum_m \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \frac{y^\ell}{\ell!} (\partial^\ell \phi_k)(\xi) \rho_m(\xi) e^{ix\cdot m} \prod_{j=1}^n U_j^{m_j} \, \mathrm{d}x \, \mathrm{d}y \\
&= \sum_{|\ell| < N_1} \frac{1}{\ell!} (\partial^\ell \phi_k)(\xi) \sum_m \rho_m(\xi) \prod_{j=1}^n U_j^{m_j} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot(y-m)} y^\ell \, \mathrm{d}x \, \mathrm{d}y \\
&= \sum_{|\ell| < N_1} \frac{1}{\ell!} (\partial^\ell \phi_k)(\xi) \sum_m \rho_m(\xi) \prod_{j=1}^n U_j^{m_j} m^\ell \\
&= \sum_{|\ell| < N_1} \frac{1}{\ell!} (\partial^\ell \phi_k)(\xi) (\delta^\ell \rho)(\xi).
\end{aligned}$$

Let $\mu(\xi) = \sum_{k=0}^{\infty} \mu_k(\xi)$. It remains to show that μ is of order $M_1 + M_2$. Obviously,

$$\sum_{N \le |\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi)(\xi)(\delta^\ell \rho)(\xi) \in S^{M_1+M_2-N},$$

so we just need to show that the remainder is of order $M_1 + M_2 - N_1$. Note that

$$\mu(\xi) - \sum_{|\ell| < N_1} \frac{y^\ell}{\ell!} (\partial^\ell \phi)(\xi)(\delta^\ell \rho)(\xi) = \sum_{k=0}^{\infty} T_{N_1}^{(k)}(\xi),$$

where

$$T_{N_1}^{(k)}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} R_{N_1}(y, \xi) \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y$$

and

$$R_{N_1}(y, \xi) = N_1 \sum_{|\ell| = N_1} \frac{y^\ell}{\ell!} \int_0^1 (1-\gamma)^{N_1 - 1} (\partial^\ell \phi_k)(\xi + \gamma y) \, \mathrm{d}\gamma.$$

Integrating by parts, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \frac{y^\ell}{\ell!} \int_0^1 (1-\gamma)^{N_1-1} (\partial^\ell \phi_k)(\xi + \gamma y) \, \mathrm{d}\gamma \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} y^\ell \int_0^1 (1-\gamma)^{N_1-1} (\partial^\ell (\phi_k)(\xi + \gamma y) \, \mathrm{d}\gamma \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \int_0^1 (1-\gamma)^{N_1-1} (\partial^\ell (\phi_k)(\xi + \gamma y) \, \mathrm{d}\gamma D_x^\ell \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \int_0^1 (1-\gamma)^{N_1-1} (\partial^\ell (\phi_k)(\xi + \gamma y) \, \mathrm{d}\gamma \delta^\ell \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{\ell!} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \int_0^1 (1-\gamma)^{N_1-1} (\partial^\ell (\phi_k)(\xi + \gamma y) \, \mathrm{d}\gamma \alpha_x((\delta^\ell \rho)(\xi)) \, \mathrm{d}x \, \mathrm{d}y,
\end{aligned}$$

where for arbitrary $a = \sum_m a_m \prod_j U_j^{m_j}$ we have

$$\begin{aligned}
(D_x)^\ell \alpha_x(a) &= (D_x)^\ell \alpha_x \left(\sum_m a_m \prod_{j=1}^n U_j^{m_j} \right) \\
&= (D_x)^\ell \sum_m a_m e^{ix\cdot m} \prod_{j=1}^n U_j^{m_j} \\
&= \sum_m a_m m^\ell e^{ix\cdot m} \prod_{j=1}^n U_j^{m_j} \\
&= \delta^\ell \alpha_x(a).
\end{aligned}$$

Since $|\ell| = N_1$, we have $\partial^\ell(\phi_k) \in S^{M_2-N_1}$ and $\delta^\ell \rho \in S^{M_1}$. We get the boundedness of

$$\mu^{N_1-M_1-M_2}(\xi) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} R_{N_1}(y,\xi) \alpha_x(\rho(\xi))\, \mathrm{d}x\, \mathrm{d}y$$

since $\delta^\ell \rho \in S^{M_1}$ and $\partial^\ell R_{N_1}(y,\xi)$ is the rest of index N_1 in Taylor's expansion of $\partial^\ell \phi_k(\xi + \gamma y)$ for which on has $\partial^\ell \phi_k \in S^{M_2-N_1}$. □

1.5 Sobolev spaces on the noncommutative n torus

We develop a theory of Sobolev spaces on the noncommutative n-torus. Sobolev spaces on noncommutative tori were originally introduced by Spera [66], but have also been written about by various other researchers [40, 52, 57, 58, 65, 70]. We construct H^s, the Sobolev spaces, in terms of the Hilbert inner product, and C^k, the analog of k times continuously differentiable functions, in terms of the C^* norm. Let $\lambda(\xi) = (1 + \xi_1^2 + \cdots + \xi_n^2)^{1/2}$. Consider the following inner product on A_θ^∞.

Definition 1.5.1. Define the Sobolev inner product $\langle \cdot, \cdot \rangle_s : A_\theta^\infty \times A_\theta^\infty \to \mathbb{C}$ by

$$\langle a, b \rangle_s := \langle P_{\lambda^s}(a), P_{\lambda^s}(b) \rangle = \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^s \overline{b_m} a_m.$$

Note that for $s = 0$ this agrees with $\langle \cdot, \cdot \rangle$. This inner product induces the following norm.

Definition 1.5.2. Define the Sobolev norm $|| \cdot ||_s : A_\theta^\infty \to \mathbb{R}_{\geq 0}$ by

$$||a||_s^2 := \langle P_{\lambda^s}(a), P_{\lambda^s}(a) \rangle = \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^s |a_m|^2.$$

Using this norm, we can define the analog of Sobolev spaces on the noncommutative n torus.

Definition 1.5.3. Define the Sobolev space H^s to be the completion of A_θ^∞ with respect to $||\cdot||_s$.

We can prove that a pseudo-differential operator of order $d \in \mathbb{R}$ continuously maps H^s into H^{s-d}. However, we must first prove the case where $s = d$.

Theorem 1.5.4. *Suppose $\rho \in S^d$. Then $||P_\rho(a)||_0 \leq C||a||_d$ for some constant $C > 0$ and P_ρ defines a bounded operator $P_\rho : H^d \to H^0$.*

Proof. Note that $\{\prod_j U_j^{m_j} : m \in \mathbb{Z}^n\}$ is an orthogonal basis with respect to $\langle\cdot,\cdot\rangle_s$, which is orthonormal in the case $s = 0$. We have $||\prod_j U_j^{m_j}||_0^2 = 1$, $||\rho_m(\xi)\prod_j U_j^{m_j}||_0^2 = |\rho_m(\xi)|^2$, and $||\rho(\xi)||_0^2 = \sum_m |\rho_m(\xi)|^2$. Since ρ is of order d, we have $||\rho(\xi)||_0 \leq C_\rho(1+|\xi|)^d$, and since $(1-|\xi|)^2 \geq 0$ gives us $(1+|\xi|)^2 \leq 2(1+|\xi|^2)$, we have

$$||\rho(\xi)||_0^2 \leq C_\rho^2(1+|\xi|)^{2d} \leq C_\rho^2 2^d(1+|\xi|^2)^d.$$

Let $k_\rho := C_\rho^2 2^d$. Then we have

$$\sum_m |\rho_m(\xi)|^2 \leq k_\rho(1+|\xi|^2)^d.$$

Let $e_{s,m} := (1+|m_1|^2+\cdots+|m_n|^2)^{-s/2}\prod_j U_j^{m_j}$ and $E_s := \{e_{s,m} \mid m \in \mathbb{Z}^n\}$. By definition we have E_s orthonormal with respect to $\langle\cdot,\cdot\rangle_s$. It suffices to prove this theorem for the case $a = e_{d,m}$ by the orthonormality of E_d since

$$||P_\rho(a)||_0^2 = \sum_m |a_m|^2||P_\rho(e_{d,m})||_0^2$$

and

$$||a||_d^2 = \sum_m |a_m|^2||e_{d,m}||_d^2.$$

Since $||e_{d,m}||_d^2 = 1$, it suffices to show that

$$||P_\rho(e_{d,m})||_0^2 \leq K$$

for some constant $K > 0$. We have

$$\begin{aligned}
||P_\rho(e_{d,m})||_0^2 &= ||\rho(m)e_{d,m}||_0^2 \\
&= ||\rho(m)(1 + |m_1|^2 + \cdots + |m_n|^2)^{-d/2} \prod_{j=1}^n U_j^{m_j}||_0^2 \\
&= \left\| \sum_k \rho_k(m) \prod_{j=1}^n U_j^{k_j} (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d/2} \prod_{j=1}^n U_j^{m_j} \right\|_0^2 \\
&= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \left\| \sum_k \rho_k(m) \prod_{j=1}^n U_j^{k_j} \prod_{j=1}^n U_j^{m_j} \right\|_0^2 \\
&= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \left\| \sum_k \rho_k(m) w(m,k) \prod_{j}^n U_j^{m_j+k_j} \right\|_0^2 \\
&= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \left\| \sum_{k,\ell} \rho_{k-m}(m) w(m,k-m) \prod_{j=1}^n U_j^{k_j} \right\|_0^2 \\
&= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \sum_k |\rho_{k-m}(m)|^2 \\
&= (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} \sum_{k,\ell} |\rho_k(m)|^2 \\
&\leq (1 + |m_1|^2 + \cdots + |m_n|^2)^{-d} k_\rho (1 + |m_1|^2 + \cdots + |m_n|^2)^d = k_\rho,
\end{aligned}$$

where

$$w(k,m) := \prod_{j=1}^n U_j^{m_j} \prod_{j=1}^n U_j^{k_j} \left(\prod_{j=1}^n U_j^{m_j+k_j} \right) \in S^1 \subset \mathbb{C}$$

so our desired constant is $K = k_\rho = C_\rho^2 2^d$ and we are done. □

For the general case $s \neq d$ we need to prove a lemma saying that $||\cdot||_s = ||\cdot||_{s-t} \circ P_{\lambda^t}$.

Lemma 1.5.5. *For any $a \in A_\theta^\infty$ and $s, t \in \mathbb{R}$, $a \in H^s$ if and only if $P_{\lambda^t}(a) \in H^{s-t}$ with $||a||_s = ||P_{\lambda^t}(a)||_{s-t}$.*

Proof. Suppose that $a \in H^s$ or $P_{\lambda^t}(a) \in H^{s-t}$. Then

$$\begin{aligned}
||P_{\lambda^t}(a)||_{s-t}^2 &= \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^{s-t} \lambda^{2t}(m) |a_m|^2 \\
&= \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^{s-t} (1 + |m_1|^2 + \cdots + |m_n|^2)^t |a_m|^2 \\
&= \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^s |a_m|^2 \\
&= ||a||_s
\end{aligned}$$

so we know that $a \in H^s$ and $P_{\lambda^t}(a) \in H^{s-t}$. □

Then the general case follows quite easily.

Corollary 1.5.6. *Suppose $\rho \in S^d$. Then $||P_\rho(a)||_{s-d} \leq C||a||_s$ for some constant $C > 0$ and P_ρ defines a bounded operator $P_\rho : H^s \to H^{s-d}$.*

Proof. By Lemma 1.5.5, we have $||P_\rho(a)||_{s-d} = ||P_{\lambda^{s-d}}(P_\rho(a))||_0$. By Theorem 1.4.1, the symbol $\sigma(P_{\lambda^{s-d}} \circ P_\rho)$ is of order $d + (s - d) = s$, so Theorem 1.5.4 gives us $||P_{\lambda^{s-d}}(P_\rho(a))||_0 \leq C||a||_s$ for some constant $C > 0$. □

We can also define an analog of the C^k norm for the noncommutative n torus.

Definition 1.5.7. Define the C^k norm $|| \cdot ||_{\infty,k} : A_\theta^\infty \to \mathbb{R}_{\geq 0}$ as follows:

$$||a||_{\infty,k} := \sum_{|\ell| \leq k} ||\delta^\ell(a)||_{C^*},$$

where the C^* norm $||\cdot||_{C^*}$ is given by

$$||a||_{C^*}^2 := \sup\{|\lambda| : a^*a - \lambda \cdot 1 \text{ not invertible}\}.$$

Since, for arbitrary $a = \sum_m a_m \prod_j U_j^{m_j}$,

$$\begin{aligned} D_s^\ell \alpha_s(a) &= (-i\partial_s)^\ell \sum_m e^{is\cdot m} a_m \prod_{j=1}^n U_j^{m_j} \\ &= \sum_m m^\ell e^{is\cdot m} a_m \prod_{j=1}^n U_j^{m_j} \\ &= \sum_m \delta^\ell e^{is\cdot m} a_m \prod_{j=1}^n U_j^{m_j} \\ &= \delta^\ell \alpha_s(a) \end{aligned}$$

we have $A_\theta^k = C^k$.

We can easily prove an analog of the Sobolev lemma as follows.

Theorem 1.5.8. *For $s > k+1$, $H^s \subseteq A_\theta^k$.*

Proof. First consider the case $k = 0$. Note that $||\cdot||_{\infty,0} = ||\cdot||_{C^*}$ so for arbitrary $a_m \prod_j U_j^{m_j}$ we have

$$||a_m \prod_{j=1}^n U_j^{m_j}||_{\infty,0}^2 = \sup\{|\lambda| : |a_m|^2 - \lambda \cdot 1 \text{ not invertible}\} = |a_m|^2$$

and for arbitrary $a = \sum_m a_m \prod_j U_j^{m_j}$ we have

$$||a||_{\infty,0}^2 \leq \sum_m ||a_m \prod_{j=1}^n U_j^{m_j}||_{\infty,0}^2 = \sum_m |a_m|^2 = ||a||_0^2$$

by the triangle inequality. We have

$$a = \sum_m a_m \lambda^s(m) \lambda^{-s}(m) \prod_{j=1}^n U_j^{m_j}$$

so by the Cauchy-Schwarz inequality we get

$$||a||_0^2 \leq ||a||_s^2 \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s}.$$

Since $2s > 2$, $(1+|m_1|^2+\cdots+|m_n|^2)^{-s}$ is summable over $m \in \mathbb{Z}^n$ and $||a||_0^2 \leq C||a||_s$.

Thus we get $||a||_{\infty,0} \leq ||a||_0 \leq C||a||_s$ and $H^s \subseteq A_\theta^0$.

Now suppose $k > 0$. Using what we've proven for the previous case, we have

$$\begin{aligned}
||\delta^\ell(a)||_{\infty,0} &\leq C||\delta^\ell(a)||_{s-|\ell|} \\
&= C|| \sum_m m^\ell a_m \prod_{j=1}^n U_j^{m_j} ||_{s-|\ell|} \\
&< C|| \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^{|\ell|} a_m \prod_{j=1}^n U_j^{m_j} ||_{s-|\ell|} \\
&= C||P_{\lambda^{|\ell|}}(a)||_{s-|\ell|} \\
&= C||a||_s
\end{aligned}$$

for $|\ell| \leq k$ since $s - |\ell| \geq s - k > 1$. Therefore,

$$||a||_{\infty,k} = \sum_{|\ell|\leq k} ||\delta^\ell(a)||_{\infty,0} \leq \sum_{|\ell|\leq k} C||a||_s \leq C||a||_s(k+1)(k+2)/2$$

and we get $H^s \subseteq A_\theta^k$. □

We get the following corollary.

Corollary 1.5.9. $\bigcap_{s\in\mathbb{R}} H^s = A_\theta^\infty$.

Proof. Suppose $a \in \bigcap_{s\in\mathbb{R}} H^s$. Then for any $k \in \mathbb{Z}_{\geq 0}$, $a \in H^{k+2}$, so by the theorem we just proved, $a \in A_\theta^k$. Consequently $a \in A_\theta^\infty$, so $\bigcap_{s\in\mathbb{R}} H^s \subseteq A_\theta^\infty$.

Suppose $a \in A_\theta^\infty$. Then since H^s is the completion of A_θ^∞ with respect to $|| \cdot ||_s$, $A_\theta^\infty \subseteq H^s$ for all $s \in \mathbb{R}$, and $A_\theta^\infty \subseteq \bigcap_{s\in\mathbb{R}} H^s$. □

We can also prove an analog of the Rellich lemma for the noncommutative n torus.

Theorem 1.5.10. *Let $\{a_N\} \in A_\theta^\infty$ be a sequence. Suppose that there is a constant C so that $||a_N||_s \leq C$ for all N. Let $s > t$. Then there is a subsequence $\{a_{N_j}\}$ that converges in H^t.*

Proof. Let $e_{s,m} := (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s/2} \prod_{j=1}^n U_j^{m_j}$ and $E_s := \{e_{s,m} \mid m \in \mathbb{Z}^n\}$. E_s is an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_s$, so we can write $a_N := \sum_k a_{N,k} e_{s,k}$. Then

$$|a_{N,k}|^2 \leq \sum_k |a_{N,k}|^2 \leq C^2$$

and $|a_{N,k}| \leq C$. Applying the Arzela-Ascoli theorem to $\{a_{N,k}\}$ for some fixed k, we can get a subsequence $\{a_{N_j,k}\}$ of $\{a_{N,k}\}$ such that for any $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that $|a_{N_i,k} - a_{N_j,k}| < \epsilon$ whenever $i, j \geq M(\epsilon)$. Do this for all $|k_1|^2 + \cdots + |k_n|^2 \leq r$, replacing $\{a_N\}$ with $\{a_{N_j}\}$ each time. Then we get a subsequence $\{a_{N_j}\}$ of $\{a_N\}$ such that for any $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that, for all $|k_1|^2 + \cdots + |k_n|^2 \leq r$, $|a_{N_i,k} - a_{N_j,k}| < \epsilon$ whenever $i, j \geq M(\epsilon)$. Now consider the sum

$$||a_{N_i} - a_{N_j}||_t^2 = \sum_k |a_{N_i,k} - a_{N_j,k}|^2 (1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s}.$$

Decompose it into two parts: one where $|k_1|^2 + \cdots + |k_n|^2 > r^2$ and one where $|k_1|^2 + \cdots + |k_n|^2 \leq r$. On $|k_1|^2 + \cdots + |k_n|^2 > r^2$ we estimate

$$(1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} < (1 + r^2)^{t-s}$$

so that

$$\sum_{|k_1|^2+\cdots+|k_n|^2\geq r^2} |a_{N_i,k} - a_{N_j,k}|^2(1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s}$$
$$< (1 + r^2)^{t-s} \sum_k |a_{N_i,k} - a_{N_j,k}|^2 \leq 2C^2(1 + r^2)^{t-s}.$$

If $\epsilon > 0$ is given, we choose r so that $2C^2(1 + r^2)^{t-s} < \epsilon$. The remaining part of the sum is over $|k_1|^2+\cdots+|k_n|^2 \leq r^2$ and can be bounded above by $\epsilon' := \epsilon - 2C^2(1+r^2)^{t-s}$ if $i, j \geq M(\sqrt[n]{\epsilon'}/(2r+1))$ because a ball of radius r centered at the origin is contained in a cube of side length $2r$ that has $(2r + 1)^n$ lattice points. Then the total sum is bounded above by ϵ, and we are done. $\square$

1.6 The pseudo-differential calculus on f.g. projective modules over the noncommutative n torus

We can generalize these results to arbitrary finitely generated projective right modules over the noncommutative n torus following p. 553 of [16], which considers finitely generated projective modules over an arbitrary unital $*$-algebra. Let E be a finitely generated projective right A_θ^∞-module. Since E is a finitely generated projective right A_θ^∞-module, we can write E as a direct summand $E = e(A_\theta^\infty)^r$ of a free module $(A_\theta^\infty)^r$ with direct complement $F = (\mathrm{id} - e)(A_\theta^\infty)^r$, where the idempotent $e \in M_r(A_\theta^\infty)$ is self-adjoint. We make frequent use of the lift $\iota : E \to (A_\theta^\infty)^r$ sending $\vec{a} \in E$ to $\vec{a} \in E \subseteq (A_\theta^\infty)^r$. Consider an $r \times r$ matrix valued symbol $\rho = (\rho_{j,k})$ where $\rho_{j,k} : \mathbb{R}^n \to A_\theta^\infty$ are scalar symbols and $\rho_{j,k} \in S^d$. Define the operator $P_\rho : E \to E$ as follows: $P_\rho(\vec{a}) := (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\rho(\xi)\alpha_s(\vec{a})\,\mathrm{d}s\,\mathrm{d}\xi$. Define the inner product $\langle \vec{a}, \vec{b}\rangle : E \times E \to \mathbb{C}$ sending $(\vec{a}, \vec{b}) \mapsto \tau(\vec{b}^*\vec{a})$. Since $\langle\cdot,\cdot\rangle_E = \langle\cdot,\cdot\rangle_{(A_\theta^\infty)^r}|_{E\times E}$, it's actually easier to work with $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{(A_\theta^\infty)^r}$, applying ι when necessary. Since $P_\rho(\vec{a})_j = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\sum_{k=1}^r \rho_{j,k}(\xi)\alpha_s(a_k)\,\mathrm{d}s\,\mathrm{d}\xi$, Lemma 1.2.5 generalizes to E as follows after applying it to each component: $P_\rho(\vec{a}) = \sum_m \rho(m)\vec{a}_m\prod_{j=1}^n U_j^{m_j}$. Theorems 1.3.1 and 1.4.1 generalize as follows.

Theorem 1.6.1. *(a) For a pseudo-differential operator P with $r\times r$ matrix valued symbol $\sigma(P) = \rho = \rho(\xi)$, the symbol of the adjoint P^* satisfies*

$$\sigma(P^*) \sim \sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq 0})^n} \frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho(\xi))^*}{\ell_1!\cdots\ell_n!}.$$

(b) If Q is a pseudo-differential operator with $r\times r$ matrix valued symbol $\sigma(Q) = \rho' = \rho'(\xi)$, then the product PQ is also a pseudo-differential operator and has symbol

$$\sigma(PQ) \sim \sum_{(\ell_1,\ldots,\ell_n)\in(\mathbb{Z}_{\geq 0})^n} \frac{\partial_1^{\ell_1}\cdots\partial_n^{\ell_n}(\rho(\xi))\delta_1^{\ell_1}\cdots\delta_n^{\ell_n}(\rho'(\xi))}{\ell_1!\cdots\ell_n!}.$$

Proof. First let's prove part (a). Let ρ be an $r\times r$ matrix valued symbol of order M and $\vec{a},\vec{b}\in E$. We have

$$\begin{aligned}\langle P_\rho(\vec{a}),\vec{b}\rangle &= \tau\left(\vec{b}^*\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\rho(\xi)\alpha_s(\vec{a})\,\mathrm{d}s\,\mathrm{d}\xi\right)\\ &= \tau\left(\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{+is\cdot\xi}\alpha_{-s}(\rho(\xi)^*\vec{b})\,\mathrm{d}s\,\mathrm{d}\xi\right)^*\vec{a}\right) = \langle\vec{a},P_\rho^*(\vec{b})\rangle,\end{aligned}$$

where

$$\begin{aligned}P_\rho^*(\vec{b})_j &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{+is\cdot\xi}\sum_{k=1}^r \alpha_{-s}(\rho(\xi)_{j,k}^* b_k)\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{+is\cdot\xi}\sum_{k=1}^r \alpha_{-s}(\rho(\xi)_{j,k}^*)\alpha_{-s}(b_k)\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \sum_{m,p}\left[\rho_m(p-m)\prod_{h=1}^n U_h^{m_h}\right]^*\left(b_p\prod_{h=1}^n U_h^{p_h}\right)\end{aligned}$$

so

$$\begin{aligned}
\sigma(P_\rho^*)(\xi) &= \left[\sum_m \rho_m(\xi - m)\prod_{h=1}^n U_h^{m_h}\right]^* \\
&= \left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\sum_m \rho_m(\xi - y)\alpha_x\left(\prod_{h=1}^n U_h^{m_h}\right)\mathrm{d}x\,\mathrm{d}y\right]^* \\
&= \left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\alpha_x(\rho(\xi - y))\,\mathrm{d}x\,\mathrm{d}y\right]^*.
\end{aligned}$$

The rest of the proof reduces to the $r = 1$ case, applying it to each entry in $\rho = (\rho_{j,k})$.

We proceed to part (b). Let ρ be an $r \times r$ matrix valued symbol of order m_1 and ϕ be an $r \times r$ matrix valued symbol of order m_2. Let $\{\varphi_k\}$ be a partition of unity and define $\phi_k(\xi) := \phi(\xi)\varphi_k(\xi)$. Let $\vec{a} \in E$. We have

$$\begin{aligned}
&P_{\phi_k}(P_\rho(\vec{a})) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\phi_k(\xi)\alpha_s(P_\rho(\vec{a}))\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi}\phi_k(\xi)\alpha_s\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-it\cdot\eta}\rho(\eta)\alpha_t(\vec{a})\,\mathrm{d}t\,\mathrm{d}\eta\right)\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-is\cdot\xi - it\cdot\eta}\phi_k(\xi)\alpha_s(\rho(\eta))\alpha_{s+t}(\vec{a})\,\mathrm{d}t\,\mathrm{d}\eta\right\}\mathrm{d}s\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi - i(y-x)\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\eta\right\}\mathrm{d}x\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot(\xi-\eta) - iy\cdot\eta}\phi_k(\xi)\alpha_x(\rho(\eta))\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\eta\right\}\mathrm{d}x\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma - iy\cdot\eta}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau\right\}\mathrm{d}x\,\mathrm{d}\sigma \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\,\mathrm{d}x\,\mathrm{d}\sigma\right\}\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\lambda_k(\tau)\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau,
\end{aligned}$$

where $\lambda_k(\tau) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\sigma}\phi_k(\sigma+\tau)\alpha_x(\rho(\tau))\,\mathrm{d}x\,\mathrm{d}\sigma$ so

$$P_\phi(P_\rho(\vec{a})) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\tau}\lambda(\tau)\alpha_y(\vec{a})\,\mathrm{d}y\,\mathrm{d}\tau,$$

where $\lambda(\tau) = \sum_{k=0}^{\infty} \lambda_k(\tau)$. Let $\lambda_k(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \phi_k(\xi + y) \alpha_x(\rho(\xi)) \, \mathrm{d}x \, \mathrm{d}y$. Since

$$\begin{aligned} \lambda_k(\xi)_{\alpha,\gamma} &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \sum_{\beta=1}^{r} \phi_k(\xi + y)_{\alpha,\beta} \alpha_x(\rho(\xi)_{\beta,\gamma}) \, \mathrm{d}x \, \mathrm{d}y \\ &= \sum_{\beta=1}^{r} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot y} \phi_k(\xi + y)_{\alpha,\beta} \alpha_x(\rho(\xi)_{\beta,\gamma}) \, \mathrm{d}x \, \mathrm{d}y, \end{aligned}$$

the rest of the proof reduces to the $r = 1$ case, applying it to each summand in the above sum. □

Let $\lambda(\xi) = (1 + \xi_1^2 + \cdots + \xi_n^2)^{1/2} \mathrm{id}_E$. Consider the following inner product on E.

Definition 1.6.2. Define the Sobolev inner product $\langle \cdot, \cdot \rangle_s : E \times E \to \mathbb{C}$ by $\langle \vec{a}, \vec{b} \rangle_s := \langle P_{\lambda^s}(\vec{a}), P_{\lambda^s}(\vec{b}) \rangle = \sum_{j,m} (1 + |m_1|^2 + \cdots + |m_n|^2)^s \overline{b_{j,m}} a_{j,m}$.

Note that for $s = 0$ this agrees with $\langle \cdot, \cdot \rangle_E$. Since $\langle \cdot, \cdot \rangle_{s,E} = \langle \cdot, \cdot \rangle_{s,(A_\theta^\infty)^r}|_{E \times E}$, it's actually easier to work with $\langle \cdot, \cdot \rangle_s := \langle \cdot, \cdot \rangle_{s,(A_\theta^\infty)^r}$, applying ι when necessary. This inner product induces the following norm.

Definition 1.6.3. Define the Sobolev norm $|| \cdot ||_{s,E} : E \to \mathbb{R}_{\geq 0}$ by

$$||\vec{a}||_{s,E}^2 := \langle P_{\lambda^s}(\vec{a}), P_{\lambda^s}(\vec{a}) \rangle = \sum_{j,m} (1 + |m_1|^2 + \cdots + |m_n|^2)^s |a_{j,m}|^2.$$

Since $|| \cdot ||_{s,E} = || \cdot ||_{s,(A_\theta^\infty)^r}|_E$, it's actually easier to work with $|| \cdot ||_s := || \cdot ||_{s,(A_\theta^\infty)^r}$, applying ι when necessary. Using this norm, we can define the analog of Sobolev spaces on E.

Definition 1.6.4. Define the Sobolev space H^s to be the completion of E with respect to $|| \cdot ||_s$.

We can prove that a pseudo-differential operator of order $d \in \mathbb{R}$ continuously maps H^s into H^{s-d}. However we must first prove the case where $s = d$.

Theorem 1.6.5. *Suppose ρ is a matrix valued symbol of order d. Then, for any $\vec{a} \in E$, $||P_\rho(\vec{a})||_0 \le C||\vec{a}||_d$ for some constant $C > 0$ and P_ρ defines a bounded operator $P_\rho : H^d \to H^0$.*

Proof. Let $\{v_j : j \in \{1, \ldots, r\}\}$ be the standard basis of $(A_\theta^\infty)^r$ considered as a finitely generated free right A_θ^∞-module. Let $\{f_j = ev_j : j \in \{1, \ldots, r\}\}$ be a generating set of the eigenvalue 1 eigenspace of e considered as an endomorphism of $(A_\theta^\infty)^r$. Note that $\{f_j \prod_g U_g^{m_g} : m \in \mathbb{Z}^n, j \in \{1, \ldots, r\}\}$ spans E as a $\mathbb{C}$-vector space, and $f_j \prod_g U_g^{m_g}$ and $f_k \prod_g U_g^{n_g}$ are orthogonal. Consider the $\mathbb{C}$-vector subspace spanned by $\{f_j \prod_g U_g^{m_g}, m \in \mathbb{Z}^n\}$ for fixed j. It has countable dimension. Let $\{f_j a_{j,m} : m \in I\}$ be a basis of this subspace with index set $I \subseteq \mathbb{Z}^n$, where $a_{j,m} \in A_\theta^\infty$. We can apply Gram–Schmidt with respect to $\langle \cdot, \cdot \rangle$ and get an orthonormal basis $F_j := \{f_{j,m} : m \in I\}$. The subspaces for different j are still orthogonal, so $F := \{f_{j,m} : m \in I, j \in \{1, \ldots, r\}\}$ is an orthogonal basis of E considered as a $\mathbb{C}$-vector space, normalized with respect to $\langle \cdot, \cdot \rangle$. We have $||f_{j,m}||_0^2 = 1$, $||\rho(\xi)_{h,j} f_{j,m}||_0^2 = ||\rho(\xi)_{h,j}||^2$, and $||\rho(\xi)_{h,j} f_j||_0^2 = \sum_h ||\rho(\xi)_{h,j}||_0^2$. Since $\rho_{h,j}$ is of order d, we have $||\rho(\xi)_{h,j}||_0 \le C_\rho(1 + |\xi|)^d$, and since $(1 - |\xi|)^2 \ge 0$ gives us $(1 + |\xi|)^2 \le 2(1 + |\xi|^2)$, we have $||\rho(\xi)_{h,j}||_0^2 \le C_\rho^2(1 + |\xi|)^{2d} \le C_\rho^2 2^d(1 + |\xi|^2)^d$. Let $k_\rho := C_\rho^2 2^d$. Then we have $||\rho(\xi)_{h,j}||_0^2 \le k_\rho(1 + |\xi|^2)^d$. Let $f_{s,j,m} := f_{j,m}/||f_{j,m}||_s$ and

$$F_s := \{f_{s,j,m} \mid m \in I, j \in \{1, \ldots, r\}\}.$$

By definition, F_s is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle_s$. It suffices to prove this theorem for the case $\vec{a} = f_{d,j,m}$ by the orthogonality of F_d since $||P_\rho(\vec{a})||_0^2 = \sum_{j,m} |a_{j,m}|^2 ||P_\rho(f_{d,j,m})||_0^2$ and $||\vec{a}||_d^2 = \sum_{j,m} |a_{j,m}|^2 ||f_{d,j,m}||_d^2$. Since $||f_{d,j,m}||_d^2 = 1$, it suffices to show that $||P_\rho(f_{d,j,m})||_0^2 \le K$ for some constant $K > 0$. We have

$$\begin{aligned}
||P_\rho(f_{d,j,m})||_0^2 = ||\rho(m) f_{d,j,m}||_0^2 &= \left\|\rho(m) f_{j,m}/||f_{j,m}||_d\right\|_0^2 \\
&= \left\|\sum_h \rho(m)_{h,j} f_{j,m}/||f_{j,m}||_d\right\|_0^2 \\
&= \frac{1}{||f_{j,m}||_d^2}\left\|\sum_h \rho(m)_{h,j} f_{j,m}\right\|_0^2 \\
&\le \frac{1}{||f_{j,m}||_d^2}\sum_h ||\rho(m)_{h,j}||_0^2 \\
&\le \frac{1}{||f_{j,m}||_d^2} r k_\rho ||f_{j,m}||_d^2 = r k_\rho
\end{aligned}$$

so our desired constant is $K = r k_\rho = r C_\rho^2 2^d$ and we are done. $\square$

For the general case $s \neq d$ we need to prove a lemma saying that $||\cdot||_s = ||\cdot||_{s-t} \circ P_{\lambda^t}$.

Lemma 1.6.6. *For any $\vec{a} \in E$ and $s, t \in \mathbb{R}$, $\vec{a} \in H^s$ if and only if $P_{\lambda^t}(\vec{a}) \in H^{s-t}$ with $||\vec{a}||_s = ||P_{\lambda^t}(\vec{a})||_{s-t}$.*

Proof. Let $\{v_j : j \in \{1, \dots, r\}\}$ be the standard basis of $(A_\theta^\infty)^r$ considered as a finitely generated free right A_θ^∞-module. Let $\{v_{j,m} = v_j \prod_g U_g^{m_g} : m \in \mathbb{Z}^n, j \in \{1, \dots, r\}\}$ be an orthonormal basis of $(A_\theta^\infty)^r$ considered as a $\mathbb{C}$-vector space. Suppose that $\vec{a} \in H^s$ or $P_{\lambda^t}(\vec{a}) \in H^{s-t}$. Then

$$||P_{\lambda^t}(\vec{a})||_{s-t}^2 = \sum_{j,m} ||v_{j,m}||_{s-t}^2 \lambda^{2t}(m) |a_{j,m}|^2 = \sum_{j,m} ||v_{j,m}||_s^2 |a_{j,m}|^2 = ||\vec{a}||_s^2$$

so we know that $\vec{a} \in H^s$ and $P_{\lambda^t}(\vec{a}) \in H^{s-t}$. $\square$

Then the general case follows quite easily.

Corollary 1.6.7. *Suppose ρ is a matrix valued symbol of order d. Then*

$$||P_\rho(\vec{a})||_{s-d} \leq C||\vec{a}||_s$$

for some constant $C > 0$ and P_ρ defines a bounded operator $P_\rho : H^s \to H^{s-d}$.

Proof. By Lemma 1.6.6, we have $||P_\rho(\vec{a})||_{s-d} = ||P_{\lambda^{s-d}}(P_\rho(\vec{a}))||_0$. By Proposition 1.6.1(b), the matrix valued symbol $\sigma(P_{\lambda^{s-d}} \circ P_\rho)$ is of order $d + (s-d) = s$, so Theorem 1.6.5 gives us $||P_{\lambda^{s-d}}(P_\rho(\vec{a}))||_0 \leq C||\vec{a}||_s$ for some constant $C > 0$. □

We can also define an analog of the C^k norm on E.

Definition 1.6.8. Define the C^k norm $||\cdot||_{\infty,k} : E \to \mathbb{R}_{\geq 0}$ as follows: $||\vec{a}||_{\infty,k} := \sum_{|\ell|\leq k} ||\delta^\ell(\vec{a})||_{C^*}$ where the C^* norm $||\cdot||_{C^*}$ is given by $||\vec{a}||_{C^*}^2 := \sup\{|\lambda| : \vec{a}^*\vec{a} - \lambda \cdot 1 \text{ not invertible}\}$.

Since, for arbitrary $\vec{a} = \sum_{j,m} f_j \prod_g U_g^{m_g} a_{j,m}$,

$$\begin{aligned} D_s^\ell \alpha_s(\vec{a}) &= (-i\partial_s)^\ell \sum_{j,m} e^{is\cdot m} f_j \prod_{g=1}^n U_g^{m_g} a_{j,m} = \sum_{j,m} m^\ell e^{is\cdot m} f_j \prod_{g=1}^n U_g^{m_g} a_{j,m} \\ &= \sum_{j,m} \delta^\ell e^{is\cdot m} f_j \prod_{g=1}^n U_g^{m_g} a_{j,m} = \delta^\ell \alpha_s(\vec{a}) \end{aligned}$$

we have $e(A_\theta^k)^r = C^k$. We can easily prove an analog of the Sobolev lemma on E as follows.

Theorem 1.6.9. *For $s > k + 1$, $H^s \subseteq e(A_\theta^k)^r$.*

Proof. First consider the case $k = 0$. Note that $||\cdot||_{\infty,0} = ||\cdot||_{C^*}$ so for arbitrary $f_j \prod_{g=1}^n U_g^{m_g} a_{j,m}$ we have $||f_j \prod_{g=1}^n U_g^{m_g} a_{j,m}||_{\infty,0}^2 = \sup\{|\lambda| : |a_{j,m}|^2 - \lambda \cdot 1 \text{ not invertible}\} = |a_{j,m}|^2$ and for arbitrary $\vec{a} = \sum_{j,m} f_j \prod_{g=1}^n U_g^{m_g} a_{j,m}$ we have $||\vec{a}||_{\infty,0}^2 \leq \sum_{j,m} ||f_j \prod_{g=1}^n U_g^{m_g} a_{j,m}||_{\infty,0}^2 \leq \sum_{j,m} |a_{j,m}|^2 = ||\vec{a}||_0^2$ by the triangle inequality. We have $\vec{a} = \sum_{j,m} \lambda^s(m) f_j \prod_{g=1}^n U_g^{m_g} a_{j,m} \lambda^{-s}(m)$ so by the Cauchy-Schwarz inequality we get $||\vec{a}||_0^2 \leq ||\vec{a}||_s^2 \sum_m (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s}$. Since $2s > 2$, $(1 + |m_1|^2 + \cdots + |m_n|^2)^{-s}$ is summable over $m \in I$ and $j \in \{1, \ldots, r\}$ so $||\vec{a}||_0^2 \leq C||\vec{a}||_s$. Thus we get $||\vec{a}||_{\infty,0} \leq ||\vec{a}||_0 \leq C||\vec{a}||_s$ and $H^s \subseteq e(A_\theta^0)^r$.

Now suppose $k > 0$. Using what we've proven for the previous case, we have

$$
\begin{aligned}
||\delta^\ell(\vec{a})||_{\infty,0} \leq C||\delta^\ell(\vec{a})||_{s-|\ell|} &= C\left\|\sum_{j,m} m^\ell f_j \prod_{g=1}^n U_g^{m_g} a_{j,m}\right\|_{s-|\ell|} \\
&< C\left\|\sum_{j,m} (1 + |m_1|^2 + \cdots + |m_n|^2)^{|\ell|} f_j \prod_{g=1}^n U_g^{m_g} a_{j,m}\right\|_{s-|\ell|} \\
&= C||P_{\lambda^{|\ell|}}(\vec{a})||_{s-|\ell|} = C||\vec{a}||_s
\end{aligned}
$$

for $|\ell| \leq k$ since $s - |\ell| \geq s - k > 1$. Therefore, $||\vec{a}||_{\infty,k} = \sum_{|\ell|\leq k} ||\delta^\ell(\vec{a})||_{\infty,0} \leq \sum_{|\ell|\leq k} C||\vec{a}||_s \leq C||\vec{a}||_s(k+1)(k+2)/2$ and we get $H^s \subseteq e(A_\theta^k)^r$. □

We get the following corollary.

Corollary 1.6.10. $\bigcap_{s\in\mathbb{R}} H^s = e(A_\theta^\infty)^r$.

Proof. Suppose $a \in \bigcap_{s\in\mathbb{R}} H^s$. Then for any $k \in \mathbb{Z}_{\geq 0}$, $\vec{a} \in H^{k+2}$, so by the theorem we just proved, $\vec{a} \in e(A_\theta^k)^r$. Consequently $\vec{a} \in e(A_\theta^\infty)^r$, so $\bigcap_{s\in\mathbb{R}} H^s \subseteq e(A_\theta^\infty)^r$.

Suppose $a \in e(A_\theta^\infty)^r$. Then since H^s is the completion of $e(A_\theta^\infty)^r$ with respect to $||\cdot||_s$, $e(A_\theta^\infty)^r \subseteq H^s$ for all $s \in \mathbb{R}$, and $e(A_\theta^\infty)^r \subseteq \bigcap_{s\in\mathbb{R}} H^s$. □

We can also prove an analog of the Rellich lemma on E.

Theorem 1.6.11. *Let $\{\vec{a}_N\} \in e(A_\theta^\infty)^r$ be a sequence. Suppose that there is a constant C so that $||\vec{a}_N||_s \leq C$ for all N. Let $s > t$. Then there is a subsequence $\{\vec{a}_{N_j}\}$ that converges in H^t.*

Proof. Let $\{v_j : j \in \{1, \dots, r\}\}$ be the standard basis of $(A_\theta^\infty)^r$ considered as a free A_θ^∞-module. Let $\{f_j = ev_j : j \in \{1, \dots, r\}\}$ be a generating set of the eigenvalue 1 eigenspace of e considered as an endomorphism of $(A_\theta^\infty)^r$. Note that $\{f_j \prod_g U_g^{m_g} : m \in \mathbb{Z}^n, j \in \{1, \dots, r\}\}$ spans E as a $\mathbb{C}$-vector space, and $f_j \prod_g U_g^{m_g}$ and $f_k \prod_g U_g^{n_g}$ are orthogonal. Consider the $\mathbb{C}$-vector subspace spanned by $\{f_j \prod_g U_g^{m_g}, m \in \mathbb{Z}^n\}$ for fixed j. It has countable dimension. Let $\{f_j a_{j,m} : m \in I\}$ be a basis of this subspace with index set $I \subseteq \mathbb{Z}^n$ where $a_{j,m} \in A_\theta^\infty$. We can apply Gram–Schmidt with respect to $\langle\cdot,\cdot\rangle$ and get an orthonormal basis $F_j := \{f_{j,m} : m \in I\}$. The subspaces for different j are still orthogonal, so $F := \{f_{j,m} : m \in I, j \in \{1, \dots, r\}\}$ is an orthogonal basis of E considered as a $\mathbb{C}$-vector space, normalized with respect to $\langle\cdot,\cdot\rangle$. Let $e_{s,j,m} := (1 + |m_1|^2 + \cdots + |m_n|^2)^{-s/2} f_{j,m}$ and

$$E_s := \{e_{s,j,m} : m \in I, j \in \{1, \dots, r\}\}.$$

By definition, E_s is an orthogonal basis with respect to $\langle\cdot,\cdot\rangle_s$, so we can write $\vec{a}_N := \sum_{h,k} e_{s,h,k} a_{N,h,k}$. Then $|a_{N,h,k}|^2 \leq \sum_{h,k} |a_{N,h,k}|^2 \leq C^2$ and $|a_{N,h,k}| \leq C$. Applying the Arzela-Ascoli theorem to $\{a_{N,h,k}\}$ for some fixed (h,k), we can get a subsequence $\{a_{N_j,h,k}\}$ of $\{a_{N,h,k}\}$ such that for any $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that $|a_{N_i,h,k} - a_{N_j,h,k}| < \epsilon$ whenever $i, j \geq M(\epsilon)$. Do this for all $g \in \{0, 1\}$, $1 \leq h \leq r$, and $|k_1|^2 + \cdots + |k_n|^2 \leq R^2$, replacing $\{a_N\}$ with $\{a_{N_j}\}$ each time. Then we get a subsequence $\{a_{N_j}\}$ of $\{a_N\}$ such that for any $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that, for all $1 \leq h \leq r$ and $|k_1|^2 + \cdots + |k_n|^2 \leq R^2$, $|a_{N_i,h,k} - a_{N_j,h,k}| < \epsilon$ whenever $i, j \geq M(\epsilon)$. Now consider the sum $||a_{N_i} - a_{N_j}||_t^2 = \sum_{h,k} |a_{N_i,h,k} - a_{N_j,h,k}|^2(1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s}$. Decompose it into two parts: one where $|k_1|^2 + \cdots + |k_n|^2 > R^2$ and one where $|k_1|^2 + \cdots + |k_n|^2 \leq R^2$. On $|k_1|^2 + \cdots + |k_n|^2 > R^2$ we estimate $(1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} < (1 + R^2)^{t-s}$ so that $\sum_h \sum_{|k_1|^2+\cdots+|k_n|^2 \geq R^2} |a_{N_i,h,k} - a_{N_j,h,k}|^2(1 + |k_1|^2 + \cdots + |k_n|^2)^{t-s} < (1 + R^2)^{t-s} \sum_{h,k} |a_{N_i,h,k} - a_{N_j,h,k}|^2 \leq 2rC^2(1 + R^2)^{t-s}$. If $\epsilon > 0$ is given, we choose R so that $2rC^2(1+R^2)^{t-s} < \epsilon$. The remaining part of the sum is over $|k_1|^2+\cdots+|k_n|^2 \leq R^2$ and can be bounded above by $\epsilon' := \epsilon - 2rC^2(1 + R^2)^{t-s}$ if $i, j \geq M(\sqrt[n]{\epsilon'}/(2R + 1))$ because an n-ball of radius R centered at the origin is contained in an n-cube of side length $2R$ that has $(2R + 1)^n$ lattice points. Then the total sum is bounded above by ϵ, and we are done. □

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Chapter 2

A TWISTED LOCAL INDEX FORMULA FOR CURVED NONCOMMUTATIVE TWO TORI

2.1 Introduction

The celebrated Atiyah-Singer index theorem provides a local formula that calculates the analytical (Fredholm) index of the Dirac operator on a compact spin manifold, with coefficients in a vector bundle, in terms of topological invariants, namely characteristic classes [1–4]. The theorem has deep applications in mathematics and theoretical physics. For instance, it implies important theorems such as the Gauss-Bonnet theorem and the Riemann-Roch theorem, and it is used in physics to count linearly the number of solutions of fundamental partial differential equations.

In noncommutative differential geometry [15, 16], an analog of the Dirac operator is used to encode the metric information and to use ideas from spectral geometry and Riemannian geometry in studying a noncommutative algebra, viewed as the algebra of functions on a space with noncommuting coordinates. That is, the data $(C^\infty(M), L^2(S), D)$ of smooth functions on a spin manifold, the Hilbert space of L^2-spinors, and the Dirac operator is extended to the notion of a *spectral triple* $(A, \mathcal{H}, D)$ where A is a noncommutative algebra acting on a Hilbert sapce $\mathcal{H}$, and D plays the role of the Dirac operator while acting in $\mathcal{H}$. The idempotents and thereby K-theory of the algebra are also used to consider the analog of vector bundles on an ordinary manifold.

The identity of the analytical index and the topological index defined by the Connes-Chern character was established in the noncommutative setting in [15]. Moreover, a local formula for the Connes-Chern character, which is suitable for explicit calculations, was derived in [11]. However, as we shall elaborate further in §2.3, the notion of a spectral triple is suitable for type II algebras in Murray-von Neumann classification of algebras, and not for type III cases. The remedy brought forth in [21] for studying the latter is the notion of a *twisted spectral triple*: they have shown that a twisted version of spectral triples can incorporate type III cases and examples that arise in noncommutative conformal geometry, and that the index pairing and the coincidence of the analytic and the topological index continues to hold in the twisted case. However, a general local formula for the index in the twisted case has not yet been found, except for the special case of twists afforded by scaling automorphism in [55]. In fact, the difficulty of the problem of finding a general local index formula for twisted spectral triples raises the need for treating more examples.

The main result of the present article is a local formula for the index of the Dirac operator $D^+_{e,\sigma}$ of a twisted spectral triple on the opposite algebra of the noncommutative two torus $\mathbb{T}^2_\theta$, which is twisted by a general noncommutative vector bundle represented by an idempotent e. The Dirac operator that we consider is associated with a general metric in the canonical conformal class on $\mathbb{T}^2_\theta$. Our approach is based on using the McKean-Singer index formula [54] and performing explicit heat kernel calculations by employing Connes' pseudodifferential calculus developed in

[13] for C^*-dynamical systems. It should be noted that, following the Gauss-Bonnet theorem proved in [23] for the canonical conformal class on $\mathbb{T}_\theta^2$ and its extension in [30] to general conformal classes, study of local geometric invariants of curved metrics on noncommutative tori has received remarkable attention in recent years [19, 22, 25, 32, 33, 49, 51]. For an overview of these developments one can refer to [34, 48].

This article is organized as follows. In §2.2 we provide background material about the noncommutative two torus, its pseudodifferential calculus, and heat kernel methods. In §2.3 we explain our construction of a vector bundle over the noncommutative two torus using a twisted spectral triple with σ-connections and derive the symbols of the operators necessary for the index calculation. In §2.4 we calculate the explicit local terms that give the index. The appendix contains proofs of new rearrangement lemmas, which overcome the new challenges posed in our calculations due to the presence of an idempotent as well as a conformal factor in our calculations in the noncommutative setting.

2.2 Preliminaries

In this section we provide some background material about the noncommutative two torus $\mathbb{T}_\theta^2$, and explain how one can use a noncommutative pseudodifferential calculus to derive the heat kernel expansion for an elliptic operator on $\mathbb{T}_\theta^2$.

Noncommutative two torus

For a fixed irrational number θ, the noncommutative two torus $\mathbb{T}_\theta^2$ is a *noncommutative manifold* whose algebra of *continuous* functions $C(\mathbb{T}_\theta^2)$ is the universal (unital) C^*-algebra generated by two unitary elements U and V that satisfy the commutation relation

$$VU = e^{2\pi i\theta} UV.$$

The ordinary two torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ acts on $C(\mathbb{T}_\theta^2)$ by

$$\alpha_s\left(U^m V^n\right) = e^{is\cdot(m,n)}\, U^m V^n, \qquad s \in \mathbb{T}^2, \qquad m,n \in \mathbb{Z}. \tag{2.1}$$

Since this action, in principle, comes from translation by $s \in \mathbb{T}^2$ written in the Fourier mode, its infinitesimal generators $\delta_1, \delta_2 : C^\infty(\mathbb{T}_\theta^2) \to C^\infty(\mathbb{T}_\theta^2)$ are the derivations that are analogs of partial differentiations on the ordinary two torus and are given by the defining relations

$$\delta_1(U) = U, \qquad \delta_1(V) = 0, \qquad \delta_2(U) = 0, \qquad \delta_2(V) = V.$$

The space $C^\infty(\mathbb{T}_\theta^2)$ of the *smooth* elements consists of all elements in $C(\mathbb{T}_\theta^2)$ that are smooth with respect to the action α given by (2.1), which turns out to be a dense subalgebra of $C(\mathbb{T}_\theta^2)$ that can alternatively be described as the space of all elements of the form $\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n$ with rapidly decaying complex coefficients $a_{m,n}$. The analog of integration is provided by the linear functional $\tau : C(\mathbb{T}_\theta^2) \to \mathbb{C}$ which is the bounded extension of the linear functional that sends any smooth element with the *noncommutative Fourier expansion* $\sum_{m,n} a_{m,n} U^m V^n$ to its constant term $a_{0,0}$. The functional τ turns out to be a positive trace on $C(\mathbb{T}_\theta^2)$ and we shall view it as the volume form of a flat canonical metric on $\mathbb{T}_\theta^2$. In §2.3, we will explain how one can consider a general metric in the conformal class of the flat canonical metric on $\mathbb{T}_\theta^2$ by means of a positive invertible element e^{-h}, where h is a self-adjoint element in $C^\infty(\mathbb{T}_\theta^2)$.

Heat kernel expansion

We will explain in §2.3 that our method of calculation of a noncommutative local index formula, which is the main result of this article, is based on the McKean-Singer index formula [54] and calculation of the relevant terms in small time heat kernel expansions. Here we elaborate on the derivation of the small heat kernel expansion for an elliptic differential operator of order 2 on $C(\mathbb{T}_\theta^2)$. The main tool that will be used is the pseudodifferential calculus developed in [13] for C^*-dynamical systems (see also [41, 67]). This calculus, in the case of $\mathbb{T}_\theta^2$, associates to a *pseudodifferential symbol* $\rho : \mathbb{R}^2 \to C^\infty(\mathbb{T}_\theta^2)$ a pseudodifferential operator $P_\rho : C^\infty(\mathbb{T}_\theta^2) \to C^\infty(\mathbb{T}_\theta^2)$ by the formula

$$P_\rho(a) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is\cdot\xi} \rho(\xi)\, \alpha_s(a)\, ds\, d\xi, \qquad a \in C^\infty(\mathbb{T}_\theta^2).$$

For example, any differential operator given by a finite sum of the form $\sum a_{j_1,j_2} \delta_1^{j_1} \delta_2^{j_2}$ with $a_{j_1,j_2} \in C^\infty(\mathbb{T}_\theta^2)$ is associated with the polynomial $\sum a_{j_1,j_2} \xi_1^{j_1} \xi_2^{j_2}$. In general a smooth map $\rho : \mathbb{R}^2 \to C^\infty(\mathbb{T}_\theta^2)$ is a pseudodifferential symbol of order $m \in \mathbb{Z}$ if for any non-negative integers i_1, i_2, j_1, j_2 there is a constant C such that

$$||\partial_1^{j_1} \partial_2^{j_2} \delta_1^{i_1} \delta_2^{i_2} \rho(\xi)|| \leq C(1 + |\xi|)^{m-j_1-j_2},$$

where ∂_1 and ∂_2 are respectively the partial differentiations with respect to the coordinates ξ_1 and ξ_2 of $\xi \in \mathbb{R}^2$. We denote the space of symbols of order m by S^m. A symbol ρ of order m is *elliptic* if $\rho(\xi)$ is invertible for larger enough ξ and there is a constant c such that

$$||\rho(\xi)^{-1}|| \leq c(1 + |\xi|)^{-m}.$$

In the case of differential operators, one can see that the symbol is elliptic if its leading part is invertible away from the origin.

An important feature of an elliptic operator is that it admits a *parametrix*, namely an inverse in the algebra of pseudodifferential operators modulo infinitely smoothing operators. The latter are the operators whose symbols belong to the intersection of all symbols $S^{-\infty} = \cap_{m \in \mathbb{Z}} S^m$. This illuminates the importance of pseudodifferential calculus for solving elliptic differential equations.

For our purposes in this article, it is crucial to illustrate that given a positive elliptic differential operator $\triangle$ of order 2 on $C^\infty(\mathbb{T}^2_\theta)$, one can use the pseudodifferential calculus to derive a small time asymptotic expansion for the trace of the heat kernel $\exp(-t\triangle)$. Note that the pseudodifferential symbol of $\triangle$ is of the form

$$\sigma_\triangle(\xi) = p_2(\xi) + p_1(\xi) + p_0(\xi),$$

where each p_k is homogeneous of order k in ξ. Since the eigenvalues of $\triangle$ are real and non-negative, using the Cauchy integral formula and a clockwise contour γ that goes around the non-negative real line, one can write

$$\exp(-t\triangle) = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} (\triangle - \lambda)^{-1} \, d\lambda. \tag{2.2}$$

Since $\triangle - \lambda$ is an elliptic operator, its parametrix R_λ will approximate $(\triangle - \lambda)^{-1}$ appearing above in the integrand. Since $\triangle - \lambda$ is of order 2, one can write for the symbol of R_λ:

$$\sigma_{R_\lambda}(\xi, \lambda, \triangle) = b_0(\xi, \lambda, \triangle) + b_1(\xi, \lambda, \triangle) + b_2(\xi, \lambda, \triangle) + \cdots ,$$

where each r_j is of order $-2 - j$.

Then, by considering the ellipticity of $\triangle - \lambda$ and the following composition rule, which gives an asymptotic expansion for the symbol of the composition of two pseudodifferential operators,

$$\sigma_{P_1 P_2} \sim \sum_{i_1, i_2 \in \mathbb{Z}_{\geq 0}} \frac{1}{i_1! i_2!} \partial_1^{i_1} \partial_2^{i_2} \sigma_{P_1} \, \delta_1^{i_1} \delta_2^{i_2} \sigma_{P_2},$$

one can find a recursive formula for the r_j. That is, it turns out that

$$b_0(\xi, \lambda, \triangle) = (p_2(\xi) - \lambda)^{-1}, \tag{2.3}$$

and for any $n \in \mathbb{Z}_{\geq 1}$,

$$b_n(\xi, \lambda, \triangle) = -\left(\sum_{\substack{i_1 + i_2 + j + 2 - k = n \\ 0 \leq j < n, 0 \leq k \leq 2}} \frac{1}{i_1! i_2!} \partial_1^{i_1} \partial_2^{i_2} b_j(\xi, \lambda, \triangle) \, \delta_1^{i_1} \delta_2^{i_2} (p_k(\xi)) \right) b_0(\xi, \lambda, \triangle). \tag{2.4}$$

By calculating these terms, one can replace $(\triangle - \lambda)^{-1}$ in (2.2) by R_λ and find an approximation of $\exp(-t\triangle)$.

Then, calculating the trace of this approximation, one can derive a small time asymptotic expansion for $\mathrm{Tr}(\exp(-t\triangle))$. That is, one finds that there are elements

$$a_{2n}(\triangle) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_\gamma e^{-\lambda} b_{2n}(\xi, \lambda, \triangle) \, d\lambda \, d\xi \in C^\infty(\mathbb{T}_\theta^2) \tag{2.5}$$

such that as $t \longrightarrow 0^+$, for any $a \in C^\infty(\mathbb{T}^2_\theta)$:

$$\mathrm{Tr}(a \exp(-t\triangle)) \sim t^{-1} \sum_{n=0}^{\infty} \tau\left(a\, a_{2n}(\triangle)\right)\, t^n. \tag{2.6}$$

The terms $a_{2n}(\triangle)$ are local geometric invariants associated with the operator $\triangle$ when it is a natural geometric operator such as a Laplacian, or the square of a twisted Dirac operator as in this article.

2.3 Noncommutative geometric spaces and index theory

Spectral triples

Geometric spaces are defined in terms of spectral data in noncommutative geometry [16]. A noncommutative geometric space is a *spectral triple* $(A, \mathcal{H}, D)$, where A is an involutive algebra represented by bounded operators on a Hilbert space $\mathcal{H}$, and D is the analog of the Dirac operator. That is, D is an unbounded self-adjoint operator on its domain which is dense in the Hilbert space $\mathcal{H}$, and the spectrum of D has similar properties to the spectrum of the Dirac operator on a compact Riemannian spin^c manifold, or even more generally, that of an elliptic self-adjoint differential operator of order 1 on a compact manifold. The *spectral dimension* of such a triple is the smallest positive real number d such that $|D|^{-d}$ is in the domain of the *Dixmier trace* Tr_ω [26] (or one can say that for any $\epsilon > 0$, the operator $|D|^{-d-\epsilon}$ is a trace-class operator which means that its eigenvalues are summable). Moreover, an important assumption is that the commutator of D and the action of any $a \in A$ on $\mathcal{H}$, $[D, a] := Da - aD$, extends to a bounded operator on $\mathcal{H}$. In this paradigm, the algebra A is allowed to be noncommutative, viewed as the algebra of functions on a space with noncommuting coordinates, and the metric information is encoded in the operator D.

The main classical example of this setup is the triple $\big(C^\infty(M), L^2(S), D_g\big)$, where $C^\infty(M)$ is the algebra of smooth complex-valued functions on a compact Riemannian manifold with a spin^c structure S, $L^2(S)$ is the Hilbert space of the L^2-spinors, and D_g is the Dirac operator associated with the metric, which acts on its domain in $L^2(S)$ and squares to a Laplace-type operator, see for example [5] for details about the Dirac operator. Since S is a vector bundle over M, the algebra $C^\infty(M)$ acts naturally by bounded operators on the Hilbert space $L^2(S)$, namely: $(f \cdot s)(x) := f(x)s(x), f \in C^\infty(M), s \in L^2(S), x \in M$. More importantly, the latter action interacts in a bounded manner with the Dirac operator D_g in the sense that the commutator of D_g with the action of each $f \in C^\infty(M)$ is a bounded operator as $[D_g, f] = D_g f - f D_g = c(df)$, where $c(df)$ denotes the Clifford multiplication on S by the de Rham differential of f. By construction, the Dirac operator D_g depends heavily on the Riemannian metric g on M. It is known from classical facts in spectral geometry that the spectral dimension of the spectral triple $\big(C^\infty(M), L^2(S), D_g\big)$ is equal to the dimension of the manifold M, and that the important local curvature related information can be detected in the small time asymptotic expansions of the form

$$\mathrm{Tr}\left(f \exp(-tD_g^2)\right) \quad \sim_{t\to 0^+} \quad t^{-\dim M/2} \sum_{j=0}^{\infty} t^j \int_M f(x)\, a_{2j}(x)\, dx, \qquad f \in C^\infty(M), \tag{2.7}$$

where the densities $a_{2j}(x)\, dx$ are uniquely determined by the Riemann curvature tensor and its contractions and covariant derivatives, cf. [20].

The importance of the Dirac operator, for using the tools of Riemannian geometry in studying noncommutative algebras, is fully illustrated in [18] by showing that the Dirac operator D_g contains the full metric information. That is, it is shown that any spectral triple $(A, \mathcal{H}, D)$ whose algebra A is commutative, and satisfies suitable conditions, is equivalent to the spectral triple $\left(C^\infty(M), L^2(S), D_g\right)$ of a Riemannian spin^c manifold. In fact, following the Gauss-Bonnet theorem for the noncommutative two torus proved in [23] and its extension in [30], the calculation and conceptual understanding of the local curvature terms in noncommutative geometry has attained remarkable attention in recent years [6, 19, 22, 25, 27, 32, 33, 49–51]. In these studies, the noncommutative local geometric invariants are detected in the analogs of the small time asymptotic expansion (2.7) written for spectral triples.

Twisted spectral triples

It turns out that the notion of a spectral triple $(A, \mathcal{H}, D)$ is suitable for studying algebras that possess a non-trivial trace, and a *twisted* notion of spectral triples is proposed in [21] to incorporate algebras that do not have this property. The reason is that if $(A, \mathcal{H}, D)$ is a spectral triple with spectral dimension d, then using the Dixmier trace Tr_ω, one can define the linear function $\phi : A \to \mathbb{C}$ by $\phi(a) = \mathrm{Tr}_\omega(a|D|^{-d})$, which turns out to be a trace, under the minimal regularity assumption that the commutators $[|D|, a]$ are also bounded for any $a \in A$. The main reason for the trace property $\phi(ab) = \phi(ba), a, b \in A$, is that for any $a \in A$, the commutator $[|D|^{-d}, a] = |D|^{-d}a - a|D|^{-d}$ belongs to the ideal of compact operators on which the Dixmier trace Tr_ω vanishes. Therefore, in order to incorporate algebras coming from type III examples in the Murray-von Neumann classification of algebras, which do not possess a non-trivial trace functional, the notion of a *twisted spectral triple* was introduced in [21]. The difference is that they change the definition of a spectral triple $(A,\mathcal{H},D)$ by introducing a twist by an algebra automorphism $\sigma : A \to A$, and by requiring that instead of the ordinary commutators, the twisted commutators of the form $[D,a]_\sigma := Da - \sigma(a)D$, $a \in A$, extend to bounded operators on the Hilbert space $\mathcal{H}$. We note that it is natural to assume the mild regularity condition that the twisted commutators with $|D|$ are also bounded operators and to consider a grading: a bounded selfadjoint operator γ on $\mathcal{H}$ that squares to identity, commutes with the action of A and anti-commutes with D. The boundedness of the twisted commutators is then used to observe that for any $a \in A$, the operator $|D|^{-d}a - \sigma^{-d}(a)|D|^{-d}$ is in the kernel of the Dixmier trace (which is an ideal), hence, the linear functional $a \mapsto \mathrm{Tr}_\omega(a|D|^{-d})$, $a \in A$ is a twisted trace. That is, $\mathrm{Tr}_\omega(a\,b|D|^{-d}) = \mathrm{Tr}_\omega(b\,\sigma^{-d}(a)|D|^{-d})$, for any $a,b \in A$.

It is emphasized in [21] on the important issue about twisted spectral triples that their Connes-Chern character lands in the ordinary cyclic cohomology whose pairing with the K-theory of the algebra can be realized as the index of a Fredholm operator. That is, let $(A,\mathcal{H},D,\sigma \in \mathrm{Aut}(A))$ be a twisted spectral triple of spectral dimension d. Then by passing to the phase $F = D/|D|$ one arrives at an ordinary *Fredholm module* $(A,\mathcal{H},F)$ with the *Connes-Chern character* [14],

$$\Phi(a_0,a_1,\dots,a_d) = \mathrm{Tr}\left(\gamma F[F,a_0][F,a_1]\cdots[F,a_d]\right), \qquad a_0,a_1,\dots a_d \in A. \quad (2.8)$$

This multilinear functional is a *cyclic cocycle*, it pairs with the K-theory of A as the Fredholm index of an operator, and the result of the pairing depends only on the cyclic cohomology class of Φ and the K-theory class of any chosen idempotent [14, 21]. However, for the purpose of explicit calculations, one needs to have a local formula that is cohomologous to Φ in cyclic cohomology or equivalently in the (b, B)-bicomplex. For ordinary spectral triples (when the automorphism σ is the identity), the local formula of Connes and Moscovici provides the desired formula in the (b, B)-bicomplex [11], see also [17]. As a first step towards a local formula for twisted spectral triples, in [21] they have shown that there is a *Hochschild cocycle* associated with twisted spectral triples, which is defined by

$$\Psi(a_0, a_1, \ldots, a_d) = \mathrm{Tr}_\omega \left(\gamma a_0 [D, \sigma^{-1}(a_1)]_\sigma [D, \sigma^{-2}(a_2)]_\sigma \cdots [D, \sigma^{-d}(a_d)]_\sigma |D|^{-d} \right), \tag{2.9}$$

for $a_0, a_1, \ldots a_d \in A$.

Explicit calculations are often possible with this multi-linear map Ψ, as for example one can use the trace theorem of [10] and its extension to noncommutative tori [31] to write it in terms of local formulas. However, the issue is that Ψ is only a Hochschild coccyle and in general it is not cyclic, therefore it does not pair with the K-theory (see [29, 39] for the relation between the Hochschild cocycle and Connes-Chern character in Hochschild cohomology). Finding a twisted version of the local index formula of [11] has so far proved to be a challenging problem, and it has only been done for the special examples of scaling automorphisms in [55]. For the treatment of the problem in twisted cyclic cohomology one can refer to [9, 44, 45, 53, 62]. We elaborate further on index theoretic aspects of the present paper and its relation with classical results and results in noncommutative geometry in §2.3.

Noncommutative conformal geometry

As a motivating example it is shown in [21] that the twisted notion of a spectral triple arises naturally in noncommutative conformal geometry. They consider the fact that on a Riemannian spin manifold, the Dirac operator of the conformal perturbation $g' = e^{-4h}g$ of a metric g is unitarily equivalent to $e^h D e^h$, where D is the Dirac operator of g. Thus, starting from a spectral triple $(A, \mathcal{H}, D)$ they use a selfadjoint element $h \in A$ to encode the conformal perturbation of the metric in the operator $D' = e^h D e^h$. However, when A is noncommutative, $(A, \mathcal{H}, D')$ is not a spectral triple any more, and one needs the twist $\sigma(a) = e^{2h} a e^{-2h}$, $a \in A$, to have bounded twisted commutators $[D', a]_\sigma$.

In fact, twisted spectral triples of this nature can be constructed in a more intrinsic manner as in [23] for the noncommutative two torus with a conformally flat metric, and as in the extension of this construction in [28] to ergodic C^*-dynamical systems, see also [59]. Because of its intimate relevance to the present work, we review here the construction in [23].

Viewing the canonical normalized trace $\tau : C(\mathbb{T}^2_\theta) \to \mathbb{C}$ on the noncommutative two torus as the volume form of the flat metric, an element $h = h^* \in C^\infty(\mathbb{T}^2_\theta)$ is fixed and the positive invertible element $e^{-h} \in C^\infty(\mathbb{T}^2_\theta)$ is used as a conformal factor to perturb the flat metric conformally. The volume form of the curved metric is given by the linear functional $\varphi : C(\mathbb{T}^2_\theta) \to \mathbb{C}$ defined by $\varphi(a) = \tau(ae^{-h}), a \in C(\mathbb{T}^2_\theta)$. This linear functional turns out to be a KMS state with a 1-parameter group of automorphisms. The Dirac operator of the curved metric is constructed by using the following analogs of the Dolbeault operators:

$$\partial = \delta_1 + i\delta_2, \qquad \bar{\partial} = \delta_1 - i\delta_2.$$

For conceptual details of the notion of complex structure in noncommutative geometry we refer the reader to [16], see also [46]. Let $\mathcal{H}^+$ be the Hilbert space obtained by the GNS construction from $(C(\mathbb{T}_\theta^2), \varphi)$ and consider

$$\partial_\varphi = \partial : \mathcal{H}^+ \to \mathcal{H}^-,$$

where $\mathcal{H}^-$, the analogue of $(1,0)$-forms, is the Hilbert space completion of finite sums $\sum a\partial(b), a, b \in C^\infty(\mathbb{T}_\theta^2)$, with respect to the inner product

$$\langle a\partial b, c\partial d\rangle = \tau(c^* a(\partial b)(\partial d)^*).$$

In this setting the Dirac operator is defined on the Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ as

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

where the adjoint ∂_φ^* of ∂_φ is seen to be given by [23]

$$\partial_\varphi^*(a) = \bar{\partial}(a)k^2, \qquad \partial_\varphi^* = R_{k^2}\bar{\partial},$$

where $k = \exp(h/2)$, and R_{k^2} denotes the right multiplication by k^2. The crucial point is that the action of $C(\mathbb{T}_\theta^2)$ on $\mathcal{H}$ gives rise to an ordinary spectral triple while the action of the opposite algebra $C(\mathbb{T}_\theta^2)^{op}$ of $C(\mathbb{T}_\theta^2)$ leads to a twisted spectral triple triple [23]. The main reason for this phenomena is that the action of $C(\mathbb{T}_\theta^2)$ on $\mathcal{H}^+$ induced by left multiplication is a $*$-representation of the algebra; however, the action of $C(\mathbb{T}_\theta^2)^{op}$ on $\mathcal{H}^+$ induced by right multiplication is not a $*$-representation and requires a modification which can be provided by an algebra automorphism. That is for $a \in C(\mathbb{T}^2_\theta)^{op}$ and $\xi^+ \in \mathcal{H}^+$, one has to define

$$a^{op} \cdot \xi^+ = a_+(\xi^+) = \xi^+ k^{-1} a k,$$

hence the appearance of the automorphism of $C(\mathbb{T}^2_\theta)^{op}$ given by

$$\sigma(a^{op}) = k^{-1} a k, \qquad a \in C(\mathbb{T}^2_\theta).$$

Note that for any $\xi^- \in \mathcal{H}^-$ one stays with $a^{op} \cdot \xi^- = \xi^- a$ since the inner product on $(1,0)$-forms is not affected by the conformal perturbation. This provides the ingredients of a twisted spectral triple, namely for any $a \in C(\mathbb{T}^2_\theta)^{op}$ the twisted commutator $Da^{op} - \sigma(a^{op})D$ is a bounded operator on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (while the ordinary commutators are not necessarily bounded).

We also consider the grading $\gamma : \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \to \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ given by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and for convenience write the action of any $a \in C^\infty(\mathbb{T}^2_\theta)^{op}$ on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ as

$$\pi(a) = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix},$$

where, clearly, the operators $a^\pm : \mathcal{H}^\pm \to \mathcal{H}^\pm$ are induced by

$$a_+(\xi^+) = \xi^+ k^{-1} a k, \qquad a_+ = R_{k^{-1}ak} = R_{\sigma(a)},$$

$$a_-(\xi^-) = \xi^- a, \qquad a_- = R_a.$$

We also mention that the automorphism σ satisfies the natural condition

$$\sigma(a)^* = \sigma^{-1}(a^*), \qquad \sigma^{-1} \circ * = * \circ \sigma.$$

Dirac operator twisted by a module (vector bundle)

Twisting the Dirac operator with a hermitian vector bundle by means of a hermitian connection is a standard technique in differential geometry. In the noncommutative setting, this can be carried out by compressing the operator $F = D/|D|$ of the Fredholm module of a twisted spectral triple $(A, \mathcal{H}, D, \sigma \in \mathrm{Aut}(A))$ by an idempotent e as in [21]. However, in order to implement the twisting with the operator D, one has to consider a twisted compression [60]. That is, the index of the operator $\sigma(e)De$ is important to be calculated. Note that in general e is an idempotent matrix in $M_q(A)$ for some positive integer q, but for our purposes for the noncommutative two torus, we can consider $e \in C^\infty(\mathbb{T}^2_\theta)$, see [56, 63]. Thus, we need to consider the operator

$$\sigma(e)De = \begin{pmatrix} 0 & \sigma(e_+)\partial_\varphi^* e_- \\ \sigma(e_-)\partial_\varphi e_+ & 0 \end{pmatrix} : e\mathcal{H} \to \sigma(e)\mathcal{H},$$

and calculate a local formula for the index of the operator

$$D^+_{e,\sigma} := \sigma(e_-)\partial_\varphi e_+ : e_+\mathcal{H}^+ \to \sigma(e_-)\mathcal{H}^-. \tag{2.10}$$

We will shortly explain in §2.3 that this can be done, using the McKean-Singer index formula, by calculating the constant terms in the small time heat kernel expansions of the form (2.6) for the operators $(D^+_{e,\sigma})^* D^+_{e,\sigma}$ and $D^+_{e,\sigma}(D^+_{e,\sigma})^*$. We calculate the relevant terms in the heat expansions by employing the pseudodifferential calculus and the method illustrated in §2.2. Therefore, the first step is to calculate the pseudodifferential symbols of these operators. It will be convenient for us to use the inclusions

$$\iota_+ : e_+\mathcal{H}^+ \to \mathcal{H}^+, \qquad \iota_- : \sigma(e_-)\mathcal{H}^- \to \mathcal{H}^-,$$

and the fact that $\iota_+^* = e_+, \sigma(e_-)^* = \iota_-$.

We have:

$$D^+_{e,\sigma} = \sigma(e_-)\,\partial_\varphi\,\iota_+ : e_+\mathcal{H}^+ \xrightarrow{\iota_+} \mathcal{H}^+ \xrightarrow{\partial_\varphi} \mathcal{H}^- \xrightarrow{\sigma(e_-)} \sigma(e_-)\mathcal{H}^-.$$

The operator

$$(D^+_{e,\sigma})^* D^+_{e,\sigma} = \iota_+^*\partial_\varphi^*\sigma(e_-)^*\sigma(e_-)\partial_\varphi\iota_+ = e_+\partial_\varphi^*\iota_-\sigma(e_-)\partial_\varphi\iota_+ : e_+\mathcal{H}^+ \to e_+\mathcal{H}^+$$

is the restriction of the operator

$$L_1^+ = e_+\partial_\varphi^*\sigma(e_-)\partial_\varphi : \mathcal{H}^+ \to \mathcal{H}^+$$

to the subspace $e_+\mathcal{H}^+$, which means that we can write

$$L_1^+ = \begin{pmatrix} (D^+_{e,\sigma})^* D^+_{e,\sigma} & * \\ 0 & 0 \end{pmatrix},$$

which yields

$$\exp(-tL_1^+) = \begin{pmatrix} \exp\left(-t\left(D^+_{e,\sigma}\right)^* D^+_{e,\sigma}\right) & * \\ 0 & 0 \end{pmatrix},$$

and

$$\operatorname{Tr}\left(\exp(-tL_1^+)\right) = \operatorname{Tr}\left(\exp\left(-t\left(D^+_{e,\sigma}\right)^* D^+_{e,\sigma}\right)\right).$$

We have a unitary map $W : \mathcal{H}_0 \to \mathcal{H}^+$ defined by

$$W(a) = ak, \qquad W = R_k,$$

and use it to work with the anti-unitarily equivalent operator

$$L^+ = JW^*L_1^+WJ : \mathcal{H}_0 \to \mathcal{H}_0,$$

where the operator J is induced by the involution of $C(\mathbb{T}^2_\theta)$. We have

$$L_1^+ = R_{\sigma(e)}\, R_{k^2}\, \bar\partial\, R_{\sigma(e)}\, \partial,$$

$$W^* L_1^+ W = R_{k^{-1}}\, R_{\sigma(e)}\, R_{k^2}\, \bar\partial\, R_{\sigma(e)}\, \partial\, R_k,$$

and

$$\begin{aligned} L^+ &= J W^* L_1^+ W J \\ &= J R_{k^{-1}}\, R_{\sigma(e)}\, R_{k^2}\, \bar\partial\, R_{\sigma(e)}\, \partial\, R_k J \\ &= (J R_{k^{-1}} J)\,(J R_{\sigma(e)} J)\,(J R_{k^2} J)\,(J \bar\partial J)\,(J R_{\sigma(e)} J)\,(J \partial J)\,(J R_k J) \\ &= k^{-1}\, \sigma(e)\, k^2\, (-\partial)\, \sigma(e)\, (-\bar\partial)\, k \\ &= k^{-1}\, \sigma(e)\, k^2\, \partial\, \sigma(e)\, \bar\partial\, k. \end{aligned}$$

We can now calculate the pseudodifferential symbol of the operator L^+.

Lemma 2.3.1. *We have* $\sigma_{L^+}(\xi) = p_2^+(\xi) + p_1^+(\xi) + p_0^+(\xi)$, *where*

$$p_2^+(\xi) = k^{-1}\, \sigma(e)\, k^2\, \sigma(e)\, k\, \left(\xi_1^2 + \xi_2^2\right) = k^{-2} e k^2 e k^2 \left(\xi_1^2 + \xi_2^2\right),$$

$$\begin{aligned} p_1^+(\xi) &= k^{-1}\, \sigma(e)\, k^2 \Big(\big(2\sigma(e)\, \delta_1(k) + \delta_1(\sigma(e))k + i\delta_2(\sigma(e))k\big)\xi_1 \\ &\qquad + \big(2\sigma(e)\, \delta_2(k) + \delta_2(\sigma(e))k - i\delta_1(\sigma(e))k\big)\xi_2 \Big), \end{aligned}$$

$$\begin{aligned} p_0^+(\xi) &= k^{-1}\, \sigma(e)\, k^2 \Big(\delta_1(\sigma(e))\, \delta_1(k) + \sigma(e)\, \delta_1^2(k) + \delta_2(\sigma(e))\, \delta_2(k) + \sigma(e)\, \delta_2^2(k) \\ &\qquad + i\big(\delta_2(\sigma(e))\, \delta_1(k) - \delta_1(\sigma(e))\, \delta_2(k)\big) \Big). \end{aligned}$$

Proof. It follows from the fact that

$$L^+ = k^{-1}\, \sigma(e)\, k^2\, \partial\, \sigma(e)\, \bar\partial\, k : \mathcal{H}_0 \to \mathcal{H}_0,$$

and for any $a \in C^\infty(\mathbb{T}_\theta^2) \subset \mathcal{H}_0$ we have

$$\begin{aligned}
& \left(\partial\, \sigma(e)\, \bar{\partial}\, k\right)(a) \\
= \;& \sigma(e)k\, \delta_1^2(a) + \sigma(e)k\, \delta_2^2(a) \\
& +\sigma(e)\delta_1(k)\delta_1(a) + \delta_1(\sigma(e)k)\delta_1(a) + \sigma(e)\delta_2(k)\delta_2(a) + \delta_2(\sigma(e)k)\delta_2(a) \\
& +i\Big(\sigma(e)\delta_1(k)\delta_2(a) - \sigma(e)\delta_2(k)\delta_1(a) + \delta_2(\sigma(e)k)\delta_1(a) - \delta_1(\sigma(e)k)\delta_2(a)\Big) \\
& +\delta_1(\sigma(e)\delta_1(k))a + \delta_2(\sigma(e)\delta_2(k))a + i\Big(\delta_2(\sigma(e)\delta_1(k))a - \delta_1(\sigma(e)\delta_2(k))a\Big).
\end{aligned}$$

□

Similarly, the operator

$$D^+_{e,\sigma}(D^+_{e,\sigma})^* = \sigma(e_-)\partial_\varphi \iota_+ \iota_+^* \partial_\varphi^* \sigma(e_-)^* = \sigma(e_-)\partial_\varphi e_+ \partial_\varphi^* \iota_- : \sigma(e_-)\mathcal{H}^- \to \sigma(e_-)\mathcal{H}^-$$

is equal to the restriction of the operator

$$L_1^- = \sigma(e_-)\partial_\varphi e_+ \partial_\varphi^* : \mathcal{H}^- \to \mathcal{H}^-$$

to the subspace $\sigma(e_-)\mathcal{H}^-$. Therefore

$$L_1^- = \begin{pmatrix} D^+_{e,\sigma}(D^+_{e,\sigma})^* & * \\ 0 & 0 \end{pmatrix},$$

which yields

$$\exp(-tL_1^-) = \begin{pmatrix} \exp\left(-tD^+_{e,\sigma}\left(D^+_{e,\sigma}\right)^*\right) & * \\ 0 & 0 \end{pmatrix},$$

and

$$\mathrm{Tr}\left(\exp(-tL_1^-)\right) = \mathrm{Tr}\left(\exp\left(-tD^+_{e,\sigma}\left(D^+_{e,\sigma}\right)^*\right)\right).$$

The operator $L_1^- : \mathcal{H}^- \to \mathcal{H}^-$ is anti-unitarily equivalent to the operator

$$L^- = JL_1^- J : \mathcal{H}_0 \to \mathcal{H}_0.$$

We have

$$L_1^- = R_{\sigma(e)}\, \partial\, R_{\sigma(e)}\, R_{k^2}\, \bar{\partial},$$

and

$$\begin{aligned} L^- &= JL_1^- J \\ &= J\, R_{\sigma(e)}\, \partial\, R_{\sigma(e)}\, R_{k^2}\, \bar{\partial}\, J \\ &= (J\, R_{\sigma(e)} J)\, (J \partial J)\, (J R_{\sigma(e)} J)\, (J R_{k^2} J)\, (J \bar{\partial}\, J) \\ &= \sigma(e)\, (-\bar{\partial})\, \sigma(e)\, k^2 (-\partial) \\ &= \sigma(e)\, \bar{\partial}\, \sigma(e)\, k^2\, \partial. \end{aligned}$$

We can now present the pseudodifferential symbol of L^-.

Lemma 2.3.2. *We have $\sigma_{L^-}(\xi) = p_2^-(\xi) + p_1^-(\xi)$, where*

$$p_2^-(\xi) = \sigma(e)^2 k^2 (\xi_1^2 + \xi_2^2) = k^{-1} e k^3 (\xi_1^2 + \xi_2^2),$$

$$\begin{aligned} p_1^-(\xi) = \sigma(e) \Big(& \delta_1(\sigma(e) k^2) \xi_1 + \delta_2(\sigma(e) k^2) \xi_2 \\ & + i \big(\delta_1(\sigma(e) k^2) \xi_2 - \delta_2(\sigma(e) k^2) \xi_1 \big) \Big). \end{aligned}$$

Proof. It follows from the fact that

$$L^- = \sigma(e)\, \bar{\partial}\, \sigma(e)\, k^2 \partial : \mathcal{H}^- \to \mathcal{H}^-,$$

and for any $a \in C^\infty(\mathbb{T}_\theta^2)$ we have

$$\begin{aligned}\left(\bar\partial\, \sigma(e)\, k^2 \partial\right)(a) &= \sigma(e)k^2\delta_1^2(a) + \sigma(e)k^2\delta_2^2(a)\\ &\quad +\delta_1(\sigma(e)k^2)\delta_1(a) + \delta_2(\sigma(e)k^2)\delta_2(a)\\ &\quad -i\delta_2(\sigma(e)k^2)\delta_1(a) + i\delta_1(\sigma(e)k^2)\delta_2(a).\end{aligned}$$

□

Index theorems

We dedicate this subsection to review some fundamental techniques and results about the heat equation proof of the index theorem, which play an important role in our approach in the noncommutative setting in the present paper. For a complete account of the details one can refer to [37, 47, 64] and references therein.

We first explain the McKean-Singer index theorem [54]. Assume that M is a compact manifold of even dimension m and that V and W are hermitian vector bundles over M. Also let $P : C^\infty(V) \to C^\infty(W)$ be an elliptic differential operator from the smooth sections of V to those of W, and for simplicity assume that P is of order 1. By constructing suitable Sobolev spaces out of V and W, it is known that P extends to a bounded operator between Sobolev spaces, which is a Fredholm operator as well. However, in order to deal with the Fredholm index of P, one can safely only consider the smooth sections, since the *regularity* stemming from the ellipticity of P ensures that all the sections in the kernel of (the extension of) P are smooth, see for example [37]. Then, in order to calculate the index,

$$\mathrm{Ind}(P) = \mathrm{Dim\ Ker}(P) - \mathrm{Dim\ Ker}(P^*),$$

where P^* is the adjoint of P, one can use the heat expansion as follows. The McKean-Singer index theorem states that for any $t > 0$:

$$\mathrm{Ind}(P) = \mathrm{Trace}\,(\exp(-tP^*P)) - \mathrm{Trace}\,(\exp(-tPP^*))\,. \tag{2.11}$$

On the other hand there are small time asymptotic expansions of the form (2.7), which depend on the local symbols. That is, there are densities $a_{2j}(x, P^*P)\,dx$ and $a_{2j}(x, PP^*)\,dx$ on M obtained locally from the pseudodifferenital symbols of P^*P and PP^* such that

$$\mathrm{Trace}\,(\exp(-tP^*P)) \quad \sim_{t\to 0^+} \quad t^{-m/2}\sum_{j=0}^{\infty} t^j \int_M a_{2j}(x, P^*P)\,dx, \tag{2.12}$$

and

$$\mathrm{Trace}\,(\exp(-tPP^*)) \quad \sim_{t\to 0^+} \quad t^{-m/2}\sum_{j=0}^{\infty} t^j \int_M a_{2j}(x, PP^*)\,dx. \tag{2.13}$$

Since the index of P in (2.11) is independent of t, after writing the expansions in the right hand side using (2.12), (2.13), the only term that is independent of t turns out to match with the index, hence

$$\mathrm{Ind}(P) = \int_M \left(a_m(x, P^*P) - a_m(x, PP^*)\right)\,dx. \tag{2.14}$$

This shows that there is a local formula for the index, and the celebrated Atiyah-Sinder index theorem identifies the density that integrates to the index in terms of characteristic classes when P is an important geometric operator. For example when P is the de Rham operator $d + d^*$ mapping the even differential forms to the odd ones, the index is the Euler characteristic of the manifold, which is equal to the integral of the pfaffian of the matrix of curvature 2-forms. In general, the index theorem states that if D_E is the Dirac operator of a spin structure D with coefficients in a vector bundle E, then the $\hat{\mathcal{A}}$-class of the tangent bundle TM and the Chern character of E give the index by the formula

$$\mathrm{Ind}(D_E) = \int_M \hat{\mathcal{A}}(TM)\, ch(E). \tag{2.15}$$

For various proofs of the index theorem one can refer to [1–4, 7, 8, 16, 36, 43], and references therein.

The elliptic theory of differential operators and pseudodifferential calculus operate perfectly in noncommutative settings as developed in [12], and analyzed in further detail for noncommutative tori in [41, 42, 67]. Moreover, there is a crucial need for local index formulas for twisted spectral triples as we explained in §2.3. Therefore, in the present paper we take the heat expansion approach to find a local formula for the index of the Dirac operator of the twisted spectral triple described in §2.3 with coefficients (or twisted by) an auxiliary finitely generated projective module on the noncommutative two torus, playing the role of a general vector bundle, cf. [56, 63]. We follow closely the setup provided in [21, 60], and the work has intimate connections with [49].

2.4 Calculation of a noncommutative local formula for the index

In this section we apply the method explained in §2.3, for the calculation of a local formula for the index of a twisted Dirac operator, to the twisted Dirac operator,

$$D^+_{e,\sigma} = \sigma(e_-)\partial_\varphi e_+ : e_+\mathcal{H}^+ \to \sigma(e_-)\mathcal{H}^-,$$

of a conformally flat metric on noncommutative two torus $\mathbb{T}_\theta^2$. Based on the discussion following the McKean-Singer formula (2.11), our treatment in §2.3 of the operators $(D_{e,\sigma}^+)^* D_{e,\sigma}^+$ and $D_{e,\sigma}^+ (D_{e,\sigma}^+)^*$ to derive their anti-unitarily equivalents L^+ and L^-, and the formula (2.6) for the terms in the heat expansion, we have:

$$\mathrm{I}nd(D_{e,\sigma}^+) = \tau \left(a_2 \left(L^+ \right) - a_2 \left(L^- \right) \right). \tag{2.16}$$

Thus, our task is now to perform individually for L^+ and L^- the recursive procedure that leads to the explicit formula (2.5) and to derive explicit formulas for $a_2 \left(L^+ \right)$ and $a_2 \left(L^- \right)$.

Using the homogeneous components of the pseudodifferential symbols of $L^\pm$ presented in Lemmas 2.3.1 and 2.3.2, we perform symbolic calculations and use formula (2.4) to calculate $b_2(\xi, \lambda, L^\pm)$. This leads to lengthy expressions which we calculate with computer assistance. The next task is to use (2.5) to derive $a_2(L^\pm)$, namely

$$a_2(L^\pm) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_\gamma e^{-\lambda} b_2(\xi, \lambda, L^\pm)\, d\lambda\, d\xi.$$

Using a homogeneity argument, see [22, 23], one can avoid the contour integration in the latter formula by setting $\lambda = -1$ and multiplying the final result result by -1. More precisely, one has

$$a_2(L^\pm) = -\int_{\mathbb{R}^2} b_2(\xi, -1, L^\pm)\, d\xi. \tag{2.17}$$

Using the trace property of τ appearing in the formula for the index (2.16), after calculating $b_2(\xi, -1, L^\pm)$, we will rotate cyclically the multiplicative term $b_0(\xi, -1, L^\pm)$ appearing at the very right side of our terms and bring it to the very left side. We then carry out the integration over $\mathbb{R}^2$ by passing to the polar coordinates $\xi_1 = r\cos\theta$, $\xi_2 = r\sin\theta$. The angular integration with respect to θ, from 0 to 2π, can be done in a straight forward manner. However, the radial integration with respect to r, from 0 to ∞, poses a challenge coming from the noncommutativity. This challenge appeared in [6, 22, 23, 32] as well and was overcome by the rearrangement lemma and led to the appearance of the modular automorphism in final formulas. In this article we need an even more elaborate version of the rearrangement lemma due to the presence of an idempotent in addition to a conformal factor, as we shall see shortly.

Computation of $\tau\left(a_2\left(L^-\right)\right)$

Considering the symbol of L^- written in Lemma 2.3.2 and using the recursive formulas (2.3) and (2.4), we have

$$b_0(\xi,-1,L^-) = (\sigma(e)k^2|\xi|^2+1)^{-1},$$

and up to a right multiplication by $-b_0(\xi,-1,L^-)$, $b_2(\xi,-1,L^-)$ is the following sum of 395 terms [68]:

$$\begin{aligned}b_2(\xi,-1,L^-) = {}& -2\xi_1^2\left(b_0^2\sigma(e)k^2\right)\sigma(e)k\delta_1^2(k) + 2i\xi_1^2\left(b_0^2\sigma(e)k^2\right)\sigma(e)k\left(\delta_1\delta_2(k)\right) - \\ & 4\xi_1^2\left(b_0^2\sigma(e)k^2\right)\sigma(e)\delta_1(k)\delta_1(k) + 2i\xi_1^2\left(b_0^2\sigma(e)k^2\right)\sigma(e)\delta_1(k)\delta_2(k) - \\ & 2\xi_1^2\left(b_0^2\sigma(e)k^2\right)\sigma(e)\delta_1^2(k)k + 2i\xi_1^2\left(b_0^2\sigma(e)k^2\right)\sigma(e)\delta_2(k)\delta_1(k) + \\ & 2i\xi_1^2\left(b_0^2\sigma(e)k^2\right)\sigma(e)\left(\delta_1\delta_2(k)\right)k - \xi_1^2\left(b_0^2\sigma(e)k^2\right)\delta_1^2(\sigma(e)k^2) - \cdots\end{aligned}$$

Passing to polar coordinates and integrating the angular variable, up to a right multiplication by $-2\pi b_0(\xi,-1,L^-)$, we get the following sum of 82 terms [68]:

$$-2\left(\sigma(e)k^2b_0^2\right)\delta_1(\sigma(e)k^2)\left(\sigma(e)k^2b_0^2\right)\delta_1(\sigma(e)k^2)r^6 -$$
$$2\left(\sigma(e)k^2b_0^2\right)\delta_2(\sigma(e)k^2)\left(\sigma(e)k^2b_0^2\right)\delta_2(\sigma(e)k^2)r^6 -$$
$$4\left((\sigma(e)k^2)^2b_0^3\right)\delta_1(\sigma(e)k^2)b_0\delta_1(\sigma(e)k^2)r^6 -$$
$$4\left((\sigma(e)k^2)^2b_0^3\right)\delta_2(\sigma(e)k^2)b_0\delta_2(\sigma(e)k^2)r^6 + \cdots,$$

The next step is to perform the integration against $r\,\mathrm{d}r$ over 0 to ∞.

An important integration formula we later prove requires that the leftmost powers of b_0 have a factor of $\sigma(e)$ to their right. Leftmost b_0's raised only to the power 1 already have this factor $\sigma(e)$ immediately to their right, and leftmost b_0's raised to powers 2 or 3 have a power of $\sigma(e)k^2$ to the left which we can move to the right, again giving us $\sigma(e)$ immediately to the right. As we explained earlier, it is also useful to use the trace property of τ to permute the multiplicative factors cyclically. Thus, instead of multiplying the above sum by $-2\pi b_0(\xi, -1, L^-)$.

Terms with all b_0 on the left

As it turns out, it is difficult to evaluate improper integrals when nontrivial idempotents are involved. For instance, consider the integral $\int_0^\infty e/(1+ex)^2\,\mathrm{d}x$. One would expect that

$$\int_0^\infty \frac{e\,\mathrm{d}x}{(1+ex)^2} = \left[\frac{-1}{1+ex}\right]_0^\infty = 1,$$

but that implies that

$$e = e \cdot 1 = e \cdot \int_0^\infty \frac{e\,\mathrm{d}x}{(1+ex)^2} = \int_0^\infty \frac{e\,\mathrm{d}x}{(1+ex)^2} = 1,$$

giving us a contradiction. Since

$$e(1 - ez + (ez)^2 - \cdots)^2 = e(1 - z + z^2 - \cdots)^2$$

for $|z| < 1$, we have

$$\frac{e}{(1 + ez)^2} = \frac{e}{(1 + z)^2},$$

so we can properly evaluate the aforementioned improper integral as follows:

$$\int_0^\infty \frac{e\,\mathrm{d}x}{(1 + ex)^2} = e\int_0^\infty \frac{\mathrm{d}x}{(1 + x)^2} = e\left[\frac{-1}{1 + x}\right]_0^\infty = e.$$

To calculate the contribution of terms with all b_0 on the left to the trace, we need a formula for

$$\int_0^\infty (\sigma(e)k^2u + 1)^{m+2}\backslash u^m \sigma(e)\,\mathrm{d}u.$$

Since $\sigma(e)$ and k do not necessarily commute, we cannot do the trick we did above. We need to reduce to the case of terms with b_0 in the middle. The trace property allows us to write

$$\begin{aligned} & \tau\left(\int_0^\infty (\sigma(e)k^2u + 1)^{m+2}\backslash u^m \sigma(e)\rho\,\mathrm{d}u\right) \\ = \;& \tau\left(\int_0^\infty (\sigma(e)k^2u + 1)^{m+1}\backslash (u^m \sigma(e)\rho)/(\sigma(e)k^2u + 1)\,\mathrm{d}u\right). \end{aligned}$$

Even though we get what looks like a more difficult integral, we can use techniques from combinatorics, Fourier analysis, and complex analysis to evaluate it, inspired by a proof given in [23] for the case $e = 1$.

Terms with b_0^2 in the middle

As in [23], we use integration by parts so that we can write terms with b_0^2 as terms with b_0 in the middle. The main difference is that this time, $b_0 = (\sigma(e)k^2u + 1)^{-1}$, so $\partial_r(b_0) = -2\sigma(e)k^2 r b_0^2$ and instead of a term like

$$\int_0^\infty r^6(k^2 b_0^3)\delta_1(k)k(k^2 b_0^2)\delta_1(k)kr\,\mathrm{d}r,$$

we would have something like

$$\int_0^\infty r^6(\sigma(e)k^2 b_0^3)\delta_1(k)k(\sigma(e)k^2 b_0^2)\delta_1(k)kr\,\mathrm{d}r,$$

which we would replace by

$$\frac{\sigma(e)k^2}{2}\int_0^\infty \partial_r(r^6 b_0^3)\delta_1(k)kb_0\delta_1(k)k\,\mathrm{d}r.$$

Again, we are able to reduce to the case of terms with b_0 in the middle.

Terms with b_0 in the middle

For integrals involving elements of $C^\infty(\mathbb{T}_\theta^2)$ squeezed between powers of $b_0 = b_0(\xi, -1, L^-)$, we need to prove a rearrangement lemma. In the most basic case, which suffices for our needs, the idempotent appears with k^2 in the denominator of the b_0, and we need the following generalized version of the rearragement lemma [23, Lemma 6.2]:

Lemma 2.4.1. *For every element $\rho \in A_\theta^\infty$ and every non-negative integer m we have*

$$\int_0^\infty (\sigma(ek^2)u+1)^{m+1}\backslash(u^m\sigma(e)\rho)/(\sigma(ek^2)u+1)\,\mathrm{d}u = \sigma(e)D_m(k^{-(2m+2)}\sigma(e)\rho\sigma(e)),$$

where

$$D_m = \mathcal{L}_m(\Delta),$$

$$\begin{aligned}\mathcal{L}_m(u) &= \int_0^\infty \frac{x^m}{(x+1)^{m+1}}\frac{1}{xu+1}\,\mathrm{d}x \\ &= (-1)^m(u-1)^{-(m+1)}\left(\log u - \sum_{j=1}^m (-1)^{j+1}\frac{(u-1)^j}{j}\right),\end{aligned}$$

$\Delta(a) = k^{-2}ak^2$, $a \in C(\mathbb{T}_\theta^2)$, *is the modular automorphism, and* $\sigma(a) = k^{-1}ak$ *is its square root.*

Proof. The proof is provided in Appendix A.1. □

The following corollary will be useful in our calculations:

Corollary 2.4.2.

$$\int_0^\infty (\Delta(ek^2e)u + 1)^{m+1} \backslash (u^m \Delta(e)\rho)/(\Delta(ek^2e)u + 1)\, \mathrm{d}u$$
$$= \Delta(e)D_m(k^{-(2m+2)}\Delta(e)\rho\Delta(e))\Delta(e)$$

Proof. The proof is provided in Appendix A.1. □

Applying Lemma 2.4.1 and using the Leibniz rule to simplify, we find that

$$\begin{aligned}\tfrac{1}{-2\pi}\,\tau\left(a_2\left(L^-\right)\right) = \tau\Big(&\tfrac{1}{2}\sigma(e)D_1\left(k^{-4}\sigma(e)k\delta_1(k)\sigma(e)\right)\delta_1(\sigma(e)k^2) + \\ &\tfrac{1}{2}i\sigma(e)D_1\left(k^{-4}\sigma(e)k\delta_1(k)\sigma(e)\right)\delta_2(\sigma(e)k^2) - \\ &\tfrac{1}{2}i\sigma(e)D_1\left(k^{-4}\sigma(e)k\delta_2(k)\sigma(e)\right)\delta_1(\sigma(e)k^2) + \\ &\tfrac{1}{2}\sigma(e)D_1\left(k^{-4}\sigma(e)k\delta_2(k)\sigma(e)\right)\delta_2(\sigma(e)k^2) + \cdots\Big),\end{aligned}$$

a total of 82 terms [68].

Computation of $\tau\left(a_2\left(L^+\right)\right)$

In a very similar manner, starting from the symbol of the operator L^+ written in Lemma 2.3.1, we calculate a local formula for $\tau\left(a_2\left(L^+\right)\right)$. We find that $b_2(\xi, -1, L^+)$, up to right multiplication by $-b_0(\xi, -1, L^+)$, is the following sum of 232 terms [68]:

$$\left((b_0')\Delta(e)\right)k^{-1}\sigma(e)k^2\sigma(e)\delta_1^2(k)+\left((b_0')\Delta(e)\right)k^{-1}\sigma(e)k^2\sigma(e)\delta_2^2(k)+$$
$$\left((b_0')\Delta(e)\right)k^{-1}\sigma(e)k^2\delta_1(\sigma(e))\delta_1(k)-i\left((b_0')\Delta(e)\right)k^{-1}\sigma(e)k^2\delta_1(\sigma(e))\delta_2(k)+$$
$$i\left((b_0')\Delta(e)\right)k^{-1}\sigma(e)k^2\delta_2(\sigma(e))\delta_1(k)+\left((b_0')\Delta(e)\right)k^{-1}\sigma(e)k^2\delta_2(\sigma(e))\delta_2(k)-$$
$$4\xi_1^2\left((b_0')^2\Delta(e)k^2\right)k^{-1}\sigma(e)k^2\sigma(e)\delta_1^2(k)-\cdots.$$

After passing to the polar coordinates and performing the angular integration, up to right multiplication by $-2\pi b_0(\xi,-1,L^+)$, we get the following sum of 82 terms [68]:

$$-2\left(\Delta(e)k^2(b_0')^2\Delta(e)\right)\delta_1(\Delta(e)k^2\Delta(e))\left(\Delta(e)k^2(b_0')^2\Delta(e)\right)\delta_1(\Delta(e)k^2\Delta(e))r^6-$$
$$2\left(\Delta(e)k^2(b_0')^2\Delta(e)\right)\delta_2(\Delta(e)k^2\Delta(e))\left(\Delta(e)k^2(b_0')^2\Delta(e)\right)\delta_2(\Delta(e)k^2\Delta(e))r^6-$$
$$4\left((\Delta(e)k^2)^2(b_0')^3\Delta(e)\right)\delta_1(\Delta(e)k^2\Delta(e))\left((b_0')\Delta(e)\right)\delta_1(\Delta(e)k^2\Delta(e))r^6-$$
$$4\left((\Delta(e)k^2)^2(b_0')^3\Delta(e)\right)\delta_2(\Delta(e)k^2\Delta(e))\left((b_0')\Delta(e)\right)\delta_2(\Delta(e)k^2\Delta(e))r^6+\cdots$$

Applying Corollary 2.4.2, which allows replacing $b_0=b_0(\xi,-1,L^+)$ with $b_0\Delta(e)$, we find that

$$\frac{1}{-2\pi}\tau\left(a_2\left(L^+\right)\right)=\tau\Big(\Delta(e)D_1\left(k^{-4}\Delta(e)k\sigma(e)\delta_1(k)\Delta(e)\right)\Delta(e)\delta_1(\Delta(e)k^2\Delta(e))+$$
$$\Delta(e)D_1\left(k^{-4}\Delta(e)k\sigma(e)\delta_2(k)\Delta(e)\right)\Delta(e)\delta_2(\Delta(e)k^2\Delta(e))+$$
$$\tfrac{1}{2}\Delta(e)D_1\left(k^{-4}\Delta(e)k\delta_1(\sigma(e))k\Delta(e)\right)\Delta(e)\delta_1(\Delta(e)k^2\Delta(e))-$$
$$\tfrac{1}{2}i\Delta(e)D_1\left(k^{-4}\Delta(e)k\delta_1(\sigma(e))k\Delta(e)\right)\Delta(e)\delta_2(\Delta(e)k^2\Delta(e))+\cdots\Big),$$

a total of 82 terms [68].

Reduction to the flat metric and the Connes-Chern number

Having calculated explicit local formulas for $\tau\left(a_2(L^\pm)\right)$, based on (2.16), we have found a local formula for the index of the twisted Dirac operator $D^+_{e,\sigma}$ on $\mathbb{T}^2_\theta$ given by (2.10), where the Dirac operator is associated with a conformally flat metric and the twisting is carried out by an idempotent playing the role of a general vector bundle. In forthcoming work, we will present a simplification of the local formula and will elaborate on the relation between properties of the functions of the modular automorphism in the simplified form and stability of the index under perturbations that leave the Dirac operator in the same connected component of Fredholm operators. We end this article by considering the case of the canonical flat metric on $\mathbb{T}^2_\theta$, which corresponds to the trivial conformal factor $k = 1$ whose corresponding modular automorphism Δ is the identity. Therefore, our formula for the index, in this case, reduces to a much simpler form, as one can replace the functions of the modular automorphism with the values of the functions at 1. This yields, for $k = 1$ [68]:

$$\operatorname{Ind} D^+_{e,\sigma} = 2\pi i\, \tau \left(e\delta_1(e)\delta_2(e) - e\delta_2(e)\delta_1(e)\right).$$

Note that

$$c_1(e) := 2\pi i\, \tau(e\delta_1(e)\delta_2(e) - e\delta_2(e)\delta_1(e))$$

is the Connes-Chern number of our projection e, and that, as proven in [24], one can construct e as a self-dual or anti-self dual projection with $c_1(e) = n$ for any $n \in \mathbb{Z}$.

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Chapter 3

SUMMARY CONCLUSION

The goal of my first project was to rederive Connes' pseudodifferential calculus on noncommutative tori with more analytic rigor, taking into account convergence issues with oscillatory integrals, expanding on the Sobolev space theory, and extending the calculus to vector bundles over noncommutative tori. This objective was achieved, and the result was a self-contained but rigorous introduction to the subject. Hyunsu Ha and Gihyun Lee, recent PhD students of Raphaël Ponge, wrote their entire theses on this pseudo-differential calculus, so their treatment is very detailed and thorough [41, 42]. They go through the Gelfand–Naimark–Segal construction, an approach not considered here.

The goal of my second project was to prove an index theorem for vector bundles over the noncommutative two torus. This objective was achieved, and it was made tractable to a large extent thanks to my familiarity with Hadamard products from analytic combinatorics, a subject I studied as an undergrad. Previous results in the literature only work for trivial vector bundles, so this is the first nontrivial result.

Appendix A

REARRANGEMENT LEMMAS

A.1 Proof of Lemma 2.4.1 and Corollary 2.4.2

The *Hadamard product* of two power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is defined by

$$(f \odot g)(z) = \sum_{n=0}^{\infty} f_n g_n z^n.$$

Proposition A.1.1. *[35, p. 424] Suppose f and g are analytic functions on a domain $D \subset \mathbb{C}$, and let $\gamma \subset D \cap (zD^{-1})$ be a closed contour. Then the Hadamard product of f and g obeys the following identity:*

$$(f \odot g)(z) = \frac{1}{2\pi i} \int_{\gamma} f(w)\, g(z/w)\, dw/w.$$

Lemma A.1.2. *Let N be an integer. For every element $\rho \in C^{\infty}(\mathbb{T}_{\theta}^{2})$ and every non-negative integer m we have*

$$\int_0^{\infty} (\sigma^N(ek^2)u + 1)^{m+1} \backslash (u^m \sigma^N(e)\rho)/(\sigma^N(ek^2)u + 1)\, \mathrm{d}u$$
$$= \sigma^N(e) D_m(k^{-(2m+2)} \sigma^N(e)\rho\sigma^N(e)),$$

where the modified logarithm $D_m = \mathcal{L}_m(\Delta)$ of the modular automorphism $\Delta(a) = k^{-2}ak^2$, $a \in C(\mathbb{T}_{\theta}^{2})$, with the square root $\sigma(a) = k^{-1}ak$, is defined via the function

$$\begin{aligned}\mathcal{L}_m(u) &= \int_0^{\infty} \frac{x^m}{(x+1)^{m+1}} \frac{1}{xu+1}\, \mathrm{d}x \\ &= (-1)^m (u-1)^{-(m+1)} \left(\log u - \sum_{j=1}^{m} (-1)^{j+1} \frac{(u-1)^j}{j} \right).\end{aligned}$$

Proof. First we look at the $N = 0$ case. We express the integrand as two Hadamard products [35, p. 424] of generating functions multiplied together, and then we make the change of variables $u = \exp(s)$. We will substitute an improper integral using

$$\frac{\exp[(h+s)/2]}{1+\exp(h+s-a-i\phi)} = \exp[i(\phi - ia)/2] \int_{-\infty}^{\infty} \frac{\exp[it(h+s-a-i\phi)]}{\exp(\pi t)+\exp(-\pi t)}\, \mathrm{d}t$$

for the closed form of the Fourier transform of the hyperbolic secant, and will express an inner integral as the Fourier transform of another function. Let a be a positive real number. We get

$$\begin{aligned}
&\int_0^\infty (ek^2u+1)^{m+1}\backslash(u^m e\rho)/(ek^2u+1)\,\mathrm{d}u \\
&= \int_0^\infty (ek^2u+1)^{m+1}\backslash(k^{2m+2}u^m(k^{-(2m+2)}e\rho))/(ek^2u+1)\,\mathrm{d}u \\
&= \int_0^\infty (k^2u)^{m+1/2}\left[\frac{1}{(1+k^2u)^{m+1}} \odot \Delta^{m+1/2}\left(\left(\frac{1}{1-u\Delta R_e}(1)\right)\right)\right] \\
&\Delta^{-1/2}(k^{-(2m+2)}e\rho)(k^2u)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{1}{1+k^2u} \odot \frac{1}{1-u\Delta R_e}(1)\right]\frac{\mathrm{d}u}{u} \\
&= \int_{-\infty}^{\infty}\left(\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp[(m+1/2)(s+h)]}{[1+\exp(h+s-a-i\theta)]^{m+1}}\right.\right. \\
&\left.\Delta^{m+1/2}\left(\left(\frac{1}{1-\Delta R_e\exp(i\theta+a)}(1)\right)\right)\,\mathrm{d}\theta\right]\cdot\Delta^{-1/2}(k^{-(2m+2)}e\rho) \\
&\left.\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp[(s+h)/2]}{1+\exp(h+s-a-i\phi)}\frac{1}{1-\Delta R_e\exp(i\phi+a)}(1)\,\mathrm{d}\phi\right]\right)\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\exp(ith)}{\exp(\pi t)+\exp(-\pi t)}\left(\int_{-\infty}^{\infty}\left[\int_0^{2\pi}\frac{\exp[(m+1/2)(s+h)]}{[1+\exp(h+s-a-i\theta)]^{m+1}}\right.\right. \\
&\left.\Delta^{m+\frac{1}{2}+it}\left(\left(\frac{1}{1-\Delta R_e\exp(i\theta+a)}(1)\right)\right)\,\mathrm{d}\theta\right]\Delta^{-\frac{1}{2}+it}(k^{-(2m+2)}e\rho) \\
&\left.\left[\int_0^{2\pi}\frac{\exp[\frac{i\phi+a}{2}+its+t(\phi-ia)]}{1-\Delta R_e\exp(i\phi+a)}(1)\,\mathrm{d}\phi\right]\mathrm{d}s\right)\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\exp(ith)}{\exp(\pi t) + \exp(-\pi t)} \left[\int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\exp[(m+1/2)(s+h)]\exp(its)}{[1+\exp(h+s-a-i\theta)]^{m+1}} \,\mathrm{d}s \right.$$
$$\left. \Delta^{m+\frac{1}{2}+it} \left(\left(\frac{1}{1-\Delta R_e \exp(i\theta + a)} (1) \right) \right) \mathrm{d}\theta \right] \Delta^{-\frac{1}{2}+it} (k^{-(2m+2)} e\rho)$$
$$\left[\int_0^{2\pi} \frac{\exp[(i\phi + a)/2 + t(\phi - ia)]}{1 - \Delta R_e \exp(i\phi + a)} (1) \,\mathrm{d}\phi \right] \mathrm{d}t.$$

For the second equality, agreement of the integrands within the radii of convergence of the geometric series implies agreement of the integrands throughout their analytic continuations:

$$\left[\sum_{n=0}^{\infty} (-ek^2u)^n \right]^{m+1} k^{2m+2} u^m (k^{-(2m+2)} e\rho) \left[\sum_{n=0}^{\infty} (-ek^2u)^n \right]$$
$$= \left[\sum_{n_1=0}^{\infty} \cdots \sum_{n_{m+1}=0}^{\infty} \prod_{q=1}^{m+1} (-k^2u)^{n_q} \left(\prod_{p=1}^{n_q} \Delta^p e \right)^* \right]$$
$$\cdot k^{2m+2} u^m (k^{-(2m+2)} e\rho) \left[\sum_{n=0}^{\infty} (-k^2u)^n \left(\prod_{p=1}^{n} \Delta^p e \right)^* \right]$$

$$= \left[\sum_{n_1=0}^{\infty} \cdots \sum_{n_{m+1}=0}^{\infty} (-k^2u)^{\sum_{q=1}^{m+1} n_q} \left(\prod_{p=1}^{\sum_{q=1}^{m+1} n_q} \Delta^p e \right)^* \right]$$
$$\cdot k^{2m+2} u^m (k^{-(2m+2)} e\rho) \left[\sum_{n=0}^{\infty} (-k^2u)^n \left(\prod_{p=1}^{n} \Delta^p e \right)^* \right].$$

Since

$$\int_{-\infty}^{\infty} \frac{\exp[(m+\frac{1}{2})(s+h)]\exp(its)}{[1+\exp(h+s-a-i\theta)]^{m+1}} \,\mathrm{d}s$$
$$= \exp[i(m+\frac{1}{2})(\theta - ai)] \int_{-\infty}^{\infty} \frac{\exp[(m+\frac{1}{2})(h+s-a-i\theta)]\exp(its)}{[1+\exp(h+s-a-i\theta)]^{m+1}} \,\mathrm{d}s$$
$$= \exp[i(m+\frac{1}{2})(\theta - ai)] \exp[-it(h-a-i\theta)] \hat{f}_m(t),$$

where $\hat{f}_m(t)$ is the Fourier transform of the function

$$f_m(s) = \frac{\exp[(m+1/2)s]}{[\exp(s)+1]^{m+1}},$$

we have

$$\int_0^\infty (ek^2u+1)^{m+1}\backslash(u^m e\rho)/(ek^2u+1)\,\mathrm{d}u$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\hat{f}_m(t)}{\exp(\pi t)+\exp(-\pi t)}$$
$$\Delta^{m+\frac{1}{2}+it}\left[\left(\left(\int_0^{2\pi}\frac{\exp[i(m+1/2+it)(\theta-ai)]}{1-\Delta R_e\exp(i\theta+a)}\,\mathrm{d}\theta\right)(1)\right)\right]$$

$$\Delta^{-\frac{1}{2}+it}(k^{-(2m+2)}e\rho)\left[\left(\int_0^{2\pi}\frac{\exp[i(1/2-it)(\phi-ia)]}{1-\Delta R_e\exp(i\phi+a)}\,\mathrm{d}\phi\right)(1)\right]\,\mathrm{d}t$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\hat{f}_m(t)}{\exp(\pi t)+\exp(-\pi t)}$$
$$\left[\left(\left(\int_0^{2\pi}\frac{\exp[i(m+1/2+it)(\nabla+\theta-ai)]}{1-\Delta R_e\exp(i\theta+a)}\,\mathrm{d}\theta\right)(1)\right)\right]$$
$$\Delta^{-\frac{1}{2}+it}\left(k^{-(2m+2)}e\rho\left[\left(\int_0^{2\pi}\frac{\exp[i(1/2-it)(\nabla+\phi-ia)]}{1-\Delta R_e\exp(i\phi+a)}\,\mathrm{d}\phi\right)(1)\right]\right)\,\mathrm{d}t.$$

For $e=1$, the inner integrals evaluate to 2π, so what we get agrees with the previous result. For $0\neq w\in\mathbb{C}\backslash(-\infty,0]$ and $a>-\ln|w|$, we integrate over a Hankel contour $H_{\epsilon,R}$ with branch cut along the negative x-axis and get

$$\int_0^{2\pi}\frac{\exp[i(m+1/2+it)(\nabla+\theta-ai)]}{1-w\exp(i\theta+a)}\,\mathrm{d}\theta$$
$$= \int_{C_{\exp a}}\frac{(\Delta z)^{m+1/2+it}}{1-zw}\frac{1}{i}\frac{\mathrm{d}z}{z}$$

$$= \frac{1}{i}\lim_{R\to\infty}\lim_{\epsilon\to0}\int_{H_{\epsilon,R}}\frac{(\exp[(m-1/2+it)(\log z+\nabla)])}{1-zw}\,\mathrm{d}z\,\Delta$$
$$= 2\pi(\exp[(m-1/2+it)\log(w^{-1}\Delta)])\Delta$$
$$= 2\pi((\Delta^{-1}w)^{\frac{1}{2}-m-it})\Delta,$$

and

$$\begin{aligned}
&\int_0^{2\pi} \frac{\exp[i(1/2 - it)(\nabla + \phi - ai)]}{1 - w\exp(i\phi + a)}\,\mathrm{d}\phi \\
&= \int_{C_{\exp a}} \frac{(\Delta z)^{1/2-it}}{1 - zw}\frac{1}{i}\frac{\mathrm{d}z}{z} \\
&= \frac{1}{i}\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_{H_{\epsilon,R}} \frac{(\exp[(-1/2 - it)(\log z + \nabla)])}{1 - zw}\,\mathrm{d}z\,\Delta \\
&= 2\pi(\exp[(-1/2 - it)\log(w^{-1}\Delta)])\Delta \\
&= 2\pi(\Delta^{-1}w)^{\frac{1}{2}+it}\Delta.
\end{aligned}$$

Since R_e is an idempotent operator, we can reduce to the previous result as follows:

$$\begin{aligned}
&\int_0^{\infty} (ek^2u + 1)^{m+1}\backslash(u^m e\rho)/(ek^2u + 1)\,\mathrm{d}u \\
&= \int_{-\infty}^{\infty} \frac{\hat{f}_m(t)(\Delta^{-1}\Delta R_e)^{\frac{1}{2}-m-it}\Delta(1)\Delta^{-\frac{1}{2}+it}(k^{-(2m+2)}e\rho(\Delta^{-1}\Delta R_e)^{\frac{1}{2}+it}\Delta(1))\,\mathrm{d}t}{\exp(\pi t) + \exp(-\pi t)} \\
&= \int_{-\infty}^{\infty} \frac{\hat{f}_m(t)}{\exp(\pi t) + \exp(-\pi t)} R_e(1)\Delta^{-\frac{1}{2}+it}(k^{-(2m+2)}e\rho R_e(1))\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\hat{f}_m(t)}{\exp(\pi t) + \exp(-\pi t)} e\Delta^{-\frac{1}{2}+it}(k^{-(2m+2)}e\rho e)\,\mathrm{d}t \\
&= e\int_{-\infty}^{\infty} \frac{\hat{f}_m(t)}{\exp(\pi t) + \exp(-\pi t)} \Delta^{-\frac{1}{2}+it}(k^{-(2m+2)}e\rho e)\,\mathrm{d}t \\
&= eD_m(k^{-(2m+2)}e\rho e).
\end{aligned}$$

Now that we have proven the lemma for $N = 0$, we may replace ρ with $\sigma^{-N}(\rho)$ and apply σ^N to both sides, which proves the lemma for any integer N. □

Corollary A.1.3. *Let N be an integer. Then*

$$\begin{aligned}
&\int_0^{\infty} (\sigma^N(ek^2e)u + 1)^{m+1}\backslash(u^m\sigma^N(e)\rho)/(\sigma^N(ek^2e)u + 1)\,\mathrm{d}u \\
&\qquad = \sigma^N(e)D_m(k^{-(2m+2)}\sigma^N(e)\rho\sigma^N(e))\sigma^N(e)
\end{aligned}$$

Proof. The $N = 0$ case of lemma we just proved implies that, for any $\rho \in A_\theta^\infty$ and every non-negative integer m, we have

$$\int_0^\infty (ek^2u + 1)^{m+1} \backslash (u^m e\rho)/(ek^2u + 1)\, \mathrm{d}u = eD_m(k^{-(2m+2)} e\rho e)$$

and

$$\int_0^\infty (ek^2u + 1)^{m+1} \backslash (u^m e\rho e)/(ek^2u + 1)\, \mathrm{d}u = eD_m(k^{-(2m+2)} e\rho e).$$

Since $1/(ek^2u+1) - e/(ek^2u+1) = 1-e$, subtracting the two equations immediately above gives us

$$\int_0^\infty (ek^2u + 1)^{m+1} \backslash (u^m e\rho(1 - e))\, \mathrm{d}u = 0.$$

Also,

$$\int_0^\infty (ek^2u + 1)^{-(m+1)} u^m e\rho(ek^2u + 1)^{-1} e\, \mathrm{d}u = eD_m(k^{-(2m+2)} e\rho e)e.$$

Since $1/(ek^2eu + 1) = 1 - e + (ek^2u + 1)\backslash e$, adding the two equations immediately above gives us

$$\int_0^\infty (ek^2u + 1)^{m+1} \backslash (u^m e\rho)/(ek^2eu + 1)\, \mathrm{d}u = eD_m(k^{-(2m+2)} e\rho e)e.$$

Since $(ek^2u + 1)^{m+1} \backslash e = (ek^2eu + 1)^{m+1} \backslash e$, we get

$$\int_0^\infty (ek^2eu + 1)^{m+1} \backslash (u^m e\rho)/(ek^2eu + 1)\, \mathrm{d}u = eD_m(k^{-(2m+2)} e\rho e)e.$$

After replacing ρ with $\sigma^{-N}(\rho)$ and applying σ^N to both sides, we are done. $\square$

We also state a corollary of our rearrangement lemma regarding orthogonal projections. Suppose $e_1, e_2 \in A_\theta^\infty$ are orthogonal self-adjoint idempotents, i.e. $e_1^2 = e_1 = e_1^*$, $e_2^2 = e_2 = e_2^*$, and $e_1e_2 = e_2e_1 = 0$. Then $(e_1+e_2)^2 = e_1^2+e_2^2 = e_1+e_2 = (e_1+e_2)^*$, so $e_1 + e_2$ is a self-adjoint idempotent and the following corollary follows from our rearrangement lemma:

Corollary A.1.4. *For every element $\rho \in A_\theta^\infty$ and every non-negative integer m we have*

$$\int_0^\infty (\sigma^N(e_1+e_2)k^2u+1)^{m+1} \backslash (u^m \sigma^N(e_1+e_2)\rho)/(\sigma^N(e_1+e_2)k^2u+1)\,\mathrm{d}u$$
$$= \sigma^N(e_1+e_2)D_m(k^{-(2m+2)}\sigma^N(e_1+e_2)\rho\sigma^N(e_1+e_2)).$$

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